

Notes on Straub and Werning

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Judd 1985 economy

Two types of agents: capitalists, who don't work + workers who can't save

Initial capital stock k_0

Inelastic labor supply

No government bonds => period by period budget constraint

Constant govt. spending g (=> always need to raise revenue)

Planner's instruments: capital taxes, and lump-sum transfers to / from workers

Planner's objective

$$\max_{\Sigma_{t=0}^{\infty}} \beta^t \{u(c_t) + \gamma U(C_t)\}$$

where c_t is consumption of the workers, and C_t is consumption of the capitalists.

Will focus later on special case $\gamma = 0$

Workers make no choices:

$$c_t = w_t + T_t$$

Capitalists choose saving / investment

$$\max_{\{C_t, a_{t+1}\}} \Sigma_{t=0}^{\infty} \beta^t U(C_t)$$

s.t.

$$C_t + a_{t+1} = R_t a_t$$

FOC

$$U'(C_{t-1}) = \beta R_t U'(C_t)$$

where

$$\begin{aligned} R_t &= (1 - tax_t) R_t^* \\ R_t^* &= 1 + r_t \end{aligned}$$

The convention is that tax_t is a tax that applies to the gross return to saving (can translate it back into a tax on the net return)

Firms solve static problems

$$\max_{k_t, l_t} \{f(k_t, l_t) - r_t k_t - w_t l_t\}$$

FOCs

$$\begin{aligned} f_k(k_t) &= r_t \\ f_l(l_t) &= w_t \end{aligned}$$

Market clearing

$$\begin{aligned} l_t &= 1 \\ a_t &= k_t \end{aligned}$$

which implies

$$w_t = (1 - \theta)f(k_t) = f(k_t) - f'(k_t)k_t$$

GBC

$$\begin{aligned} g + T_t &= (R_t^* - R_t)k_t \\ &= tax_t R_t^* k_t \end{aligned}$$

Use BC + mkt clearing to substitute out for R_{t+1} in FOC => Implementability constraints

$$U'(C_{t-1})k_t = \beta(C_t + k_{t+1})U'(C_t)$$

Note that there is one implementability constraint per period here, since there is no government debt

Implementability constraints plus resource constraint

$$c_t + C_t + g + k_{t+1} = f(k_t) + (1 - \delta)k_t$$

summarize all the equilibrium conditions.

[Why is this so? IMC embeds capitalist's budget constraint and FOC. If planner's choices are consistent with those that capitalist wants to choose, and are resource feasible, then they can be implemented, because workers make no choices.]

Planner's problem

$$\begin{aligned} \max_{\{c_t, C_t, k_{t+1}\}} \sum_{t=0}^{\infty} \beta^t \{u(c_t) + \gamma U(C_t)\} \\ \beta^t \lambda_t : c_t + C_t + g + k_{t+1} \leq f(k_t) + (1 - \delta)k_t \\ \beta^t \mu_t : \beta U'(C_t)(C_t + k_{t+1}) = U'(C_{t-1})k_t \end{aligned}$$

FOCs wrt c_t, C_t, k_{t+1}

$$\begin{aligned} u'(c_t) - \lambda_t &= 0 \\ \gamma U'(C_t) - \lambda_t + \mu_t \beta U''(C_t)(C_t + k_{t+1}) + \mu_t \beta U'(C_t) - \mu_{t+1} \beta U''(C_t) k_{t+1} &= 0 \\ -\lambda_t + \beta \lambda_{t+1} (f'(k_t) + (1 - \delta)) + \mu_t \beta U'(C_t) - \beta \mu_{t+1} U'(C_t) &= 0 \end{aligned}$$

Now note from the first and third FOCs that if the optimal policy converges to an interior solution with constant and finite values for allocations and multipliers, then it immediate that in that steady state $tax_t = 0$. To get that result we can ignore the second FOC altogether, and thus also ignore the value for γ .

That is the solution Judd conjectured. But is it the right solution? We will see that it is not

Consider the log utility special case

Here the IMC simplifies, because we can guess and verify the form of the optimal saving rule for capitalists:

Guess

$$k_{t+1} = \alpha R_t k_t$$

So

$$\begin{aligned} C_t &= R_t k_t - k_{t+1} \\ &= (1 - \alpha) R_t k_t \end{aligned}$$

Implies

$$\frac{C_{t+1}}{C_t} = \frac{(1 - \alpha) R_{t+1} (\alpha R_t k_t)}{(1 - \alpha) R_t k_t} = R_{t+1} \alpha$$

Is the capitalist FOC satisfied? Yes, iff $\alpha = \beta$!

$$k_{t+1} = \beta R_t k_t$$

So now we can combine the budget constraint and the (effective) FOC for the capitalist

$$\begin{aligned} C_t + k_{t+1} &= R_t k_t \\ k_{t+1} &= \beta R_t k_t \end{aligned}$$

to get the following simple implementability constraint

$$C_t = \frac{1 - \beta}{\beta} k_{t+1}$$

This in turn can be substituted directly into the resource constraint to simplify the problem.

With zero concern for capitalists ($\gamma = 0$), the Ramsey problem becomes:

$$\max_{\{c_t, k_{t+1}\}} \sum_{t=0}^{\infty} \beta^t u(c_t)$$

s.t.

$$\begin{aligned}\beta^t \mu_t & : c_t + \frac{1-\beta}{\beta} k_{t+1} + k_{t+1} = f(k_t) + (1-\delta)k_t \\ \beta^t \mu_t & : c_t + \frac{1}{\beta} k_{t+1} = f(k_t) + (1-\delta)k_t\end{aligned}$$

FOC wrt c_t :

$$\mu_t = u'(c_t)$$

FOC wrt k_{t+1}

$$\begin{aligned}-\mu_t \frac{1}{\beta} & = \beta \mu_{t+1} (1 - \delta + f'(k_{t+1})) \\ u'(c_t) & = \beta^2 u'(c_{t+1}) (1 - \delta + f'(k_{t+1}))\end{aligned}$$

Compare this to the capitalist's FOC in the decentralization

$$u'(C_t) = \beta u'(C_{t+1}) R_{t+1}$$

where

$$R_{t+1} = (1 - tax_{t+1}) (1 - \delta + f'(k_{t+1}))$$

If both equations are satisfied, it must be that $tax_{t+1} = 1 - \beta$. Thus, with log utility, it is optimal to tax capital at a constant rate, contrary to the Judd result.

[We can convert this to a tax on capital income. Assuming we are in a steady state, we have

$$1 = \beta^2 (1 - \delta + f'(k_{t+1})) \Rightarrow f'(k_{t+1}) - \delta = \frac{1}{\beta^2} - 1$$

so thinking about a conventional tax on capital income, we would have

$$\begin{aligned}1 & = \beta [1 + (1 - \tau_k)(f'(k_{t+1}) - \delta)] \\ (1 - \tau_k) & = \frac{\frac{1}{\beta} - 1}{\frac{1}{\beta^2} - 1} = \frac{1 - \beta}{\frac{1}{\beta} - \beta} \\ \tau_k & = \frac{\frac{1}{\beta} - 1}{\frac{1}{\beta} - \beta}\end{aligned}$$

so if $\beta = 0.96$ we have $\tau_k = 0.51$ – which is not a small tax on capital income!]

So unless log utility is a special knife-edge case (it isn't!) something must have gone wrong in the general case when we imposed that everything converged.

Consider general CES utility

$$\begin{aligned} U(C) &= \frac{C^{1-\sigma}}{1-\sigma} \\ U'(C) &= C^{-\sigma} \\ U''(C) &= -\sigma C^{-\sigma-1} \\ U''(C)C &= -\sigma C^{-\sigma} = -\sigma U'(C) \end{aligned}$$

Let

$$\begin{aligned} \kappa_t &= \frac{k_t}{C_{t-1}} \\ v_t &= \frac{U'(C_t)}{u'(c_t)} \end{aligned}$$

and recall that $\lambda_t = u'(c_t)$.

So the second FOC from the Ramsey problem can be written as

$$\begin{aligned} \gamma U'(C_t) - \lambda_t + \mu_t \beta U''(C_t)(C_t + k_{t+1}) + \mu_t \beta U'(C_t) - \mu_{t+1} U''(C_t) k_{t+1} &= 0 \\ \gamma \frac{U'(C_t)}{u'(c_t)} - 1 - \sigma \mu_t \beta \frac{U'(C_t)}{u'(c_t)} - \sigma \mu_t \beta \frac{U'(C_t)}{u'(c_t)} \frac{k_{t+1}}{C_t} + \mu_t \beta \frac{U'(C_t)}{u'(c_t)} + \sigma \mu_{t+1} \frac{U'(C_t)}{u'(c_t)} \frac{k_{t+1}}{C_t} &= 0 \\ \gamma v_t - 1 - \sigma \mu_t \beta v_t - \sigma \mu_t \beta v_t \kappa_{t+1} + \mu_t \beta v_t + \sigma \mu_{t+1} v_t \kappa_{t+1} &= 0 \end{aligned}$$

which can be rearranged to give

$$\mu_{t+1} = \mu_t \left(\frac{\sigma - 1}{\sigma \kappa_{t+1}} + 1 \right) + \frac{(1 - \gamma v_t)}{\beta \sigma v_t \kappa_{t+1}}$$

Impose $\gamma = 0$ (don't care about capitalists)

$$\mu_{t+1} = \mu_t \left(\frac{\sigma - 1}{\sigma \kappa_{t+1}} + 1 \right) + \frac{1}{\beta \sigma v_t \kappa_{t+1}}$$

Is it possible that all allocations converge to positive finite constants? That would imply $\kappa_{t+1} = \frac{k_{t+1}}{C_t}$ converges to a finite constant κ_g (and $v_t = \frac{U'(C_t)}{u'(c_t)}$ would converge to a constant v_g).

But then, from the previous equation, μ_t must grow without bound.

And

$$\mu_{t+1} - \mu_t = \mu_t \frac{(\sigma - 1)}{\sigma \kappa_g} + \frac{1}{\beta \sigma v_g \kappa_g}$$

must also grow without bound

Now return to the FOC for capital

$$\begin{aligned}
-u'(c_t) + \beta u'(c_{t+1})(f'(k_t) + (1 - \delta)) + \mu_t U'(C_t) - \beta \mu_{t+1} U'(C_t) &= 0 \\
-1 + \beta \frac{u'(c_{t+1})}{u'(c_t)}(f'(k_t) + (1 - \delta)) + \mu_t v_t - \beta \mu_{t+1} v_t &= 0 \\
\frac{u'(c_{t+1})}{u'(c_t)}(f'(k_t) + (1 - \delta)) &= \frac{\beta \mu_{t+1} v_t - \beta \mu_t v_t + 1}{\beta} \\
&= \frac{1}{\beta} + v_t (\mu_{t+1} - \mu_t)
\end{aligned}$$

Now if c_t and k_t converge to positive constants, then so does the left hand side of this equation. But we just showed that $\mu_{t+1} - \mu_t$ is rising over time, so this FOC would not be satisfied.

So we have proven that there cannot be an equilibrium in which c_t , k_{t+1} and C_t all converge to positive constants.

So let's look for an equilibrium in which at least one of them converges to zero. We know k_{t+1} cannot go to zero (because we need to produce output). So let's assume k_{t+1} converges to a constant $k_g > 0$. It is unlikely that capitalists could be forced to consume zero when sitting on positive wealth. So let's assume C_t converges to a constant C_g .

If C_t converges to a constant, then R_t must converge to $\frac{1}{\beta}$ (and thus the tax on capital must converge also). Capitalist optimization thus requires

$$C_g + k_g = \frac{1}{\beta} k_g$$

So if anything is going to zero, it must be worker consumption c_t – even though those are the group the planner cares about!

Straub and Werning show that that is in fact the nature of the solution.

Let's look again at the FOC for capital

$$u'(c_{t+1})(f'(k_t) + (1 - \delta)) = \frac{u'(c_t)}{\beta} + U'(C_t) (\mu_{t+1} - \mu_t)$$

In particular, $u'(c_t)$ rises at the same rate over time as μ_t .

That implies that we can solve for k_g from the resource constraint, imposing:

$$\frac{1 - \beta}{\beta} k_g + 0 + k_g + g = f(k_g) + (1 - \delta) k_g$$

Given the solution for k_g we can compute the supporting capital tax rate tax_g from

$$1 = \beta(1 - tax_g)(1 + f'(k_g) - \delta)$$