Notes on Straub and Werning

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Judd 1985 economy

Two differences relative to rep agent we discussed last time

Two types of agents: capitalists, who don't work + workers who can't save

Initial capital stock k_0

Inelastic labor supply

No government bonds => period by period budget constraint

Constant spending g

Planner's instruments: capital taxes, and lump-sum transfers to / from workers

Planner's obective

$$\max \Sigma \beta^t \{u(c_t) + \gamma U(C_t)\}$$

where c_t is consumption of the workers, and C_t is consumption of the capitalists. Will focus later on special case $\gamma = 0$

Workers make no choices:

$$c_t = w_t + T_t$$

Capitalists choose saving / investment

$$\max \Sigma \beta^t U(C_t)$$

s.t.

$$C_t + a_{t+1} = R_t a_t$$

FOC

$$U'(C_{t-1}) = \beta R_t U'(C_t)$$

where

$$\begin{aligned} R_t &= (1 - tax_t) R_t^* \\ R_t^* &= 1 + r_t \end{aligned}$$

The convention is that tax_t is a tax that applies to the gross return to saving (can translate it back into a tax on the net return)

Firms solve static problems

$$\max f(k_t, l_t) - r_t k_t - w_t l_t$$

FOCs

$$\begin{aligned} f_k(k_t) &= r_t \\ f_l(l_t) &= w_t \end{aligned}$$

Market clearing

$$\begin{aligned} l_t &= 1 \\ a_t &= k_t \end{aligned}$$

which implies

$$w_t = (1-\theta) f(k_t) = f(k_t) - f'(k_t) k$$

GBC

$$\begin{aligned} g + T_t &= (R_t^* - R_t) k_t \\ &= tax_t R_t^* k_t \end{aligned}$$

Use BC + mkt clearing to substitute out for R_{t+1} in FOC => Implementability constraint

$$U'(C_{t-1}) k_t = \beta (C_t + k_{t+1}) U'(C_t)$$

Note that there is one implementability constraint per period here, since there is no government debt

Implementability plus resource constraint

$$c_t + C_t + g + k_{t+1} = f(k_t) + (1-\delta) k_t$$

summarize all the equilibrium conditions.

Why is this so? IMC embeds capitalist's budget constraint and FOC. If planner's choices are consistent with those that capitalist wants to choose, and are resource feasible, then they can be implemented, because workers make no choices.

Planner's problem

$$\max \sum_t \beta^t \{u(c_t) + \gamma U(C_t)\}$$

$$\begin{aligned} \beta^t \lambda_t &: \quad c_t + C_t + g + k_{t+1} \leq f(k_t) + (1-\delta) k_t \\ \beta^t \mu_t &: \quad \beta U'(C_t)(C_t + k_{t+1}) = U'(C_{t-1}) k_t \end{aligned}$$

$$\begin{aligned} u'(c_t) - \lambda_t &= 0 \\ \gamma U'(C_t) - \lambda_t + \mu_t \beta U''(C_t)(C_t + k_{t+1}) + \mu_t \beta U'(C_t) - \mu_{t+1} \beta U''(C_t) k_{t+1} &= 0 \\ -\lambda_t + \beta \lambda_{t+1}(f'(k_t) + (1-\delta)) + \mu_t \beta U'(C_t) - \beta \mu_{t+1} U'(C_t) &= 0 \end{aligned}$$

Now note from the first and third FOCs that if the optimal policy converges to an interior solution with constant and finite values for allocations and multipliers, then it immediate that in that steady state $tax_t = 0$. To get that result we can ignore the second FOC altogether, and thus also ignore the value for γ.

That is the solution Judd conjectured. But is it the right solution? We will see that it is not

Consider the log utility special case

Here the IMC simplifies, because we can guess and verify the form of the optimal saving rule for capitalists:

Guess

$$k_{t+1} = \alpha R_t k_t$$

So

$$\begin{aligned} C_t &= R_t k_t - k_{t+1} \\ &= (1-\alpha) R_t k_t \end{aligned}$$

Implies

$$\frac{C_{t+1}}{C_t} = \frac{(1-\alpha) R_{t+1} \alpha R_t k_t}{(1-\alpha) R_t k_t} = R_{t+1} \alpha$$

Is the capitalist FOC satisfied? Yes, if $\alpha = \beta$

So now we can combine the budget constraint and the FOC for the capitalist

$$\begin{aligned} C_t + k_{t+1} &= R k_t \\ k_{t+1} &= \beta R_t k_t \end{aligned}$$

to get the following as a single implementability constraint

$$C_t = \frac{1-\beta}{\beta} k_{t+1}$$

This in turn can be substituted into the resource constraint:

Then, with zero concern for capitalists, the Ramsey problem becomes:

$$\begin{aligned} & \max \sum \beta^t u(c_t) \\ c_t + \frac{1-\beta}{\beta} k_{t+1} + k_{t+1} &= f(k_t) + (1-\delta) k_t \end{aligned}$$

(note that implementability effectively means that the planner cannot choose k_{t+1} and C_t independently)

FOC wrt c_t :

$$\mu_t = \beta^t u'(c_t)$$

FOC wrt k_{t+1}

$$\begin{aligned} -\mu_t \frac{1}{\beta} &= \mu_{t+1}\left(1-\delta+f'(k_{t+1})\right) \\ u'(c_t) &= \beta^2 u'(c_{t+1})\left(1-\delta+f'(k_{t+1})\right) \end{aligned}$$

Compare this to the capitalist's FOC

$$u'(c_t) = \beta u'(c_{t+1}) R_{t+1}$$

where

$$R_{t+1} = \left(1 - tax_{t+1}\right)\left(1-\delta+f'(k_{t+1})\right)$$

Clearly $tax_{t+1} = 1-\beta$. Thus, with log utility, it is optimal to tax capital at a constant rate, contrary to the Judd result.

So something must have gone wrong in the general case when we imposed that everything converged.

Consider general CES utility

$$\begin{aligned} U(C) &= \frac{C^{1-\sigma}}{1-\sigma} \\ U'(C) &= C^{-\sigma} \\ U''(C) &= -\sigma C^{-\sigma-1} \\ U''(C)C &= -\sigma C^{-\sigma} = -\sigma U'(C) \end{aligned}$$

Let

$$\begin{aligned} \kappa_t &= \frac{k_t}{C_{t-1}} \\ \upsilon_t &= \frac{U'(C_t)}{u'(c_t)} \end{aligned}$$

So the second FOC from the Ramsey problem can be written as

$$\begin{aligned} \gamma U'(C_t) - \lambda_t + \mu_t \beta U''(C_t)(C_t + k_{t+1}) + \mu_t \beta U'(C_t) - \mu_{t+1} U''(C_t) k_{t+1} &= 0 \\ \gamma \frac{U'(C_t)}{u'(c_t)} - 1 - \sigma\mu_t\beta \frac{U'(C_t)}{u'(c_t)} - \sigma\mu_t\beta \frac{U'(C_t)}{u'(c_t)} \frac{k_{t+1}}{C_t} + \mu_t\beta \frac{U'(C_t)}{u'(c_t)} + \sigma\mu_{t+1} \frac{U'(C_t)}{u'(c_t)} \frac{k_{t+1}}{C_t} &= 0 \\ \gamma\upsilon_t - 1 - \sigma\mu_t\beta\upsilon_t - \sigma\mu_t\beta\upsilon_t\kappa_{t+1} + \mu_t\beta\upsilon_t + \sigma\mu_{t+1}\upsilon_t\kappa_{t+1} &= 0 \end{aligned}$$

which can be rearranged to give

$$\mu_{t+1} = \mu_t \left(\frac{\sigma - 1}{\sigma\kappa_{t+1}} + 1 \right) + \frac{(1-\gamma\upsilon_t)}{\beta\sigma\upsilon_t\kappa_{t+1}}$$

Impose $\gamma = 0$

$$\mu_{t+1} = \mu_t \left(\frac{\sigma - 1}{\sigma\kappa_{t+1}} + 1 \right) + \frac{1}{\beta\sigma\upsilon_t\kappa_{t+1}}$$

It is immediate that when $\sigma > 1$ that μ_t grows without bound. So we cannot assume that the multipliers converge to constants under the optimal policy. Is it possible that all allocations converge to positive constants while μ_t grows without bound? We will show that it is not.

In particular, if the allocations converge to constants, then $\kappa_{t+1} = \frac{k_{t+1}}{C_t}$ and $\upsilon_t = \frac{U'(C_t)}{u'(c_t)}$ must also converge. Suppose these converge to constants κ_g and υ_g. But then we would have

$$\mu_{t+1} - \mu_t = \mu_t \frac{(\sigma - 1)}{\sigma \kappa_g} + \frac{1}{\beta \sigma \upsilon_g \kappa_g}$$

It is clear from this that if μ_t is increasing over time (which we showed above must be the case) then so is $\mu_{t+1} - \mu_t$.

Now return to the FOC for capital

$$\begin{aligned}
-u'(c_t) + \beta u'(c_{t+1})(f'(k_t) + (1-\delta)) + \mu_t U'(C_t) - \beta \mu_{t+1} U'(C_t) &= 0 \\
-1 + \beta \frac{u'(c_{t+1})}{u'(c_t)}(f'(k_t) + (1-\delta)) + \mu_t \upsilon_t - \beta \mu_{t+1} \upsilon_t &= 0 \\
\frac{u'(c_{t+1})}{u'(c_t)}(f'(k_t) + (1-\delta)) &= \frac{\beta \mu_{t+1} \upsilon_t - \beta \mu_t \upsilon_t + 1}{\beta} \\
&= \frac{1}{\beta} + \upsilon_t \left(\mu_{t+1} - \mu_t\right)
\end{aligned}$$

Now if c_t and k_t converge to positive constants, then so does the left hand side of this equation. But we just showed that $\mu_{t+1} - \mu_t$ is rising over time, so this FOC would not be satisfied.

So what does happen in the long run under the optimal policy?

Let's suppose k_t converges to a non-zero finite value k_g. (capital cannot go to zero with $g > 0$ and it cannot explode with $\delta > 0$, so this seems a reasonable assumption).

Let's also assume that the capital tax rate converges to a finite value, tax_g

Then r_t and R_t must both converge to constant values, r_g and R_g, which from the capitalist budget constraint, implies that C_t must also converge to a constant given by

$$C_g + k_g = R_g k_g$$

Also, from the capitalist FOC,

$$1 = \beta R_g$$

and thus

$$C_g = \frac{1-\beta}{\beta} k_g$$

Let's look again at the FOC for capital

$$u'(c_{t+1})(f'(k_t) + (1-\delta)) = \frac{u'(c_t)}{\beta} + U'(C_t)\left(\mu_{t+1} - \mu_t\right)$$

We have already rejected the possibility that c_t, C_t and k_t converge to positive constants, so if C_t and k_t do converge to positive constants, perhaps c_t converges to zero. Straub and Werning show that that is in fact the nature of the solution. In particular, $u'(c_t)$ rises at the same rate over time as μ_t.

That implies that we can solve for k_g from the resource constraint, imposing:

$$\frac{1-\beta}{\beta} k_g + 0 + k_g + g = f(k_g) + (1-\delta)k_g$$

Given the solution for k_g we can compute the supporting capital tax rate tax_g from

$$1 = \beta(1 - tax_g)(1 + f'(k_g) - \delta)$$