Reconciling Macroeconomics and Finance for the U.S. Corporate Sector: 1929 - Present*

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Abstract

We examine how to reconcile, quantitatively, the high volatility of market valuations of U.S. corporations with the relative stability of macroeconomic quantities over the period 1929-present. We use a stochastic growth model extended to incorporate factorless income as a measurement framework to investigate this apparent tension. Macroeconomic and financial variables are measured in a consistent fashion using the Integrated Macroeconomic Accounts of the United States, which offer a unified data set for the income statement, cash flows, and balance sheet of the U.S. Corporate Sector. We argue that fluctuations in expected cash flows to firm owners have been the dominant driver of fluctuations in the market value of U.S. corporation. We show that relatively modest shocks to expected future cash flows can account for the history of corporate valuations from 1929 to 2023, without appealing to fluctuations in discount rates. Further evidence in support of this hypothesis is that payout-price ratios in our data do in fact forecast growth of future cash flows to owners of firms. In particular, they forecast changes in the fraction of corporate output flowing to owners of firms. The time path for the after-tax return to investment in capital that we infer from our model tracks the risk free interest rate fairly closely, at least in the period after World War II. In this sense, our model offers a reconciliation of volatile market valuations and stable capital output ratios.

JEL Classification Numbers: E44, G12

Key Words: Dividends, Free Cash Flow, Integrated Macroeconomic Accounts, Stock market, Tobin's Q

*Very preliminary. The views expressed herein are those of the authors and not necessarily those of the Federal Reserve Bank of Minneapolis or the Federal Reserve System. We thank Ellen McGrattan and Yueran Ma for very helpful comments. The title of this paper is intended as a tribute to the work of the Bureau of Economic Analysis and Federal Reserve Board in producing the Integrated Macroeconomic Accounts which aim to present a unified accounting a macroeconomic flows and asset valuations implied by financial markets.

1 Introduction

How can the volatile valuations of equity claims to U.S. public corporations be reconciled with the relatively smooth evolution of most macroeconomic variables observed in data from the National Income and Product Accounts (NIPA)?

The development of a unified data set known as the Integrated Macroeconomic Accounts (henceforth IMA) offers the opportunity to make progress towards an answer to this question. This IMA data set is developed as a joint project between the Bureau of Economic Analysis and the Federal Reserve Board. It combines NIPA data on macroeconomic flows and stocks with comprehensive data on financial flows and balance sheets with equity measured at market value drawn from the Financial Accounts of the United States.[1] What results from this accounting exercise is a coherent set of income statements, cash flow statements, and market value balance sheets for major sectors of the U.S. economy. In this paper, we focus on these data for the U.S. corporate sector.

We use these IMA data together with a simple variant of the stochastic growth model modified to include factorless income to provide a reconciliation of the volatility of market valuations of U.S. corporations with the relative stability of most macroeconomic aggregates over the period 1929 through 2023.

In particular, we use these IMA data and our model to address three questions.

First, do these IMA data offer a picture of the returns to claims to the U.S. corporate sector and the volatility of the valuation of the firms in that sector in line with the data on public firms available from the Center for Research in Security Prices (CRSP) and Compustat databases? In the first part of this paper, we argue that the answer to this question is yes.

Second, do these IMA data shed new light on the drivers of the volatility of the market valuation of U.S. corporations over the past 100 years in comparison with previous results found with data on public firms? Again, we argue that the answer to this question is yes. We present a simple valuation model to document that there is ample volatility in cash flows to firm owners in the IMA data to resolve the "excess volatility puzzle" of Shiller (1981) without resorting to time-varying discount rates. Moreover, in an exercise echoing prior results by Larraine and Yogo (2008), we find that Campbell-Shiller regressions conducted with these data are consistent with the hypothesis that cash flow to price ratios forecast growth in future cash flows, as they should if changing expectations of future cash flows are the primary driver of changes in aggregate valuation ratios. We find, however, that cash flow to price ratios primarily forecast growth in the share of corporate value added flowing to owners of firms rather than growth in corporate value added overall. In this sense we echo

[1] The Financial Accounts of the United States produced by the Federal Reserve were formerly known as the Flow of Funds. See Cagetti et al. (2013) for an introduction to the construction of these data.

prior results by Greenwald, Lettau, and Ludvigson (2023).

Third, are the volatile market valuations of U.S. corporations consistent with the stable ratio of the replacement cost of capital to output in the corporate sector that is observed in the data after World War II? Our answer here is a qualified yes. The key driver of volatility in valuations in our model is time variation in the current and expected cash flow to firm owners is "factorless", in that it accrues neither to labor nor to capital. Shocks to factorless income do not translate into large changes in the expected return to investing in capital. Thus, the shocks that replicate the observed large changes in valuations in our model do not translate into counterfactually high investment volatility. At the same time, our model does generate a fairly steadily downward trend in the expected return to capital in the post World War II period, of a similar magnitude to the observed decline in safe rates.

In the period from 1929 through World War II we find large movements in the ratio of measured capital to output and large movements in the model-implied returns to investment in measured capital that require further research to resolve.

The paper is organized as follows.

In Section 2, we place our work in context of a large prior literature on this topic.

In Section 3, we use the IMA data to construct measures of cash flows to firm owners and firms valuation consistent with the definition of these concepts in a standard stochastic growth model. We refer to our measure of cash flows as *Free Cash Flow* from operations and define it as gross value added less Taxes less Compensation of Employees less Investment Expenditures. We refer to our measure of firm value as *Enterprise Value* and define it as the sum of the Market Value of Equity and Liabilities less Financial Assets. We show the evolution of these series in Figure 1 and display their relationship to alternative cash flow and valuation measures, including alternative aggregates from Compustat data for public firms in further figures in that section. We then demonstrate that the realized annual returns for the U.S. corporate sector over the period 1929-present constructed using these measures of free cash flow and enterprise value look remarkably similar to measures of returns obtained from the CRSP Value-Weighted Total Stock Market Index in Table 1 and Figure 4. Based on these observations we argue that the IMA are a useful unified data set for macrofinance.

We then turn to our second question of whether these IMA data shed new light on the role of fluctuations in expected cash flows in driving the volatility of the market valuations of U.S. corporations. First, we observe that one of the most striking features of these IMA data is that the free cash flows to owners of firms are extremely volatile as a fraction of output for the U.S. corporate sector. As shown in Figure 1, cash flows to owners of U.S. corporations as a ratio to corporate output follow a pronounced U-shape over the period 1929-2023, falling from roughly 14% to 6% of after tax gross value added between 1929 and World War II and rising from 6% to roughly 14% of after-tax output again from the late 1990's to 2023. The ratio of the market value of U.S. corporations to corporate sector output follows a similar U-shaped pattern over this time period. As shown in Figure 2, it is clearly evident that a large portion of the observed swings in the market valuation of U.S. corporations over the 1929-2023 time period can be "accounted" for, in a mechanical sense, by the fall and then rise in cash flows to owners of these corporations when valued at a constant dividend-price ratio. Based on these data, it is no surprise that the U.S. stock market has boomed over the past 20 years — cash flows to owners of U.S. corporations have boomed proportionately.

At the same time, it is also clear in these data that the ratio of free cash flow to enterprise value –i.e., the growth-model-consistent dividend-price ratio– has also fluctuated a great deal over the past century (Figure 2). A full accounting of the drivers of fluctuations in the market valuation of U.S. corporations must also account for what drives aggregate ratios of cash flows to value. Is it fluctuations in investors' expectations of future cash flows? Or fluctuations in the rate of return that investors demand to hold claims against the corporate sector?

Campbell-Shiller (CS) regressions are widely used to address these questions.[2] In Section 4, we run CS regressions with IMA data and find very different results than those found with public firm data. In particular, as shown in Table 2, CS regression results with IMA data favor the hypothesis that fluctuations in investors' expectations of future cash flows drive a substantial part of observed fluctuations in aggregate dividend-price ratios. We find in particular that the payout to price ratio for the U.S. corporate sector predicts growth in the ratio of free cash flow to corporate output and not growth in corporate output.

In Section 5 we build a macrofinance model that we can use as a quantitative theoretical framework to connect standard macroeconomic flows and stock to financial measures of valuations and returns. We combine the production side of a stochastic growth model as our model of output and incomes for workers and owners of the U.S. corporate sector with an exogenous specification of the pricing kernel used to value cash flows and guide investment in measured capital.

The key modification we make relative to the most basic stochastic growth model is that we assume that firms face a time-varying wedge between total revenue and total costs that leads to a pure rent for firm owners that we refer to as *factorless income* following Karabarbounis and Neiman (2019). The valuation of this factorless income drives a gap in the model between the enterprise value of the corporate sector and the stock of measured capital held by firms in that sector. The model also features an explicit model of corporate

[2] In the literature, these regressions are typically run using index number data on price per share and dividends per share which look very different from the IMA data on aggregate cash flows and valuations. See Campbell and Shiller (1988). We discuss reasons for the discrepancy between these results with different data sets in a companion paper Atkeson, Heathcote, and Perri (2024).

and value-added taxes, which have important time-varying impacts on after-tax cash flows and valuations.

We use the model to guide our division in the data of free cash flow into a portion that is compensation to owners of the measured capital in the corporate sector and the remainder which corresponds to after-tax factorless income. Similarly, we can use the model to divide the data on enterprise value into components reflecting present values of these two sources of cash flow.

We present that decomposition of the valuation of U.S. corporations in Figure 10. What is striking here is that, in the time period from 1929 to World War II, the large fluctuations in the value of U.S. corporations in this time period appear to be accounted for primarily by fluctuations in the capital output ratio, just as one would expect in the simplest stochastic growth model. In contrast, over the long time period from World War II to the present, fluctuations the value of U.S. corporations appear to be accounted for almost entirely by fluctuations in the value of factorless income, with the measured capital to output ratio being remarkably stable.

When we partition free cash flow between income to capital and factorless income, we see that the decline in free cash flow in the early decades of our sample entirely reflects declining free cash flow to capital, while the rise in the post 2000 period entirely reflects a rise in free cash flow associated with factorless income (see Figure 8).

In Section 6 we develop formulas to value expected future factorless income. We use these formulas to ask whether it is possible to account for the level and volatility of observed market valuations of U.S. corporations based on a specification of our pricing kernel in which all risk premia are constant over time. This accounting exercise is in the spirit of the calculations in Shiller (1981).

In our valuation model, the marginal impact of a shock to the expected value of factorless income in the long-run on the current valuation of factorless income is determined by the discount rate for a claim to aggregate output of the corporate sector and not by the discount rate for a claim to corporate equity. If this discount rate is low, shocks to the long run expected value of factorless income have a powerful impact on current valuations of corporate equity, and one can therefore account for the volatility of the aggregate dividend-price ratio for U.S. corporations based on small fluctuations in investors' expectations of the long run share of factorless income.[3]

[3] Lustig, Van Nieuwerburgh, and Verdelhan (2013) argue that the price-dividend ratio for a claim to aggregate output (the inverse of the discount rate) is quite high, perhaps 50 or 80 or even more. Since a claim to future fiscal primary surpluses is similar to a claim to factorless income in that its payoff is a stochastic share of aggregate output, those such as Blanchard (2019) who argue that high levels of government debt can be sustained with expectations of small changes in future primary surpluses are implicitly arguing that the price dividend ratio for a claim to aggregate output is quite high, even infinite. See also Abel and Panageas

In Section 7 we ask whether our model can account for the smooth evolution of the stock of measured corporate capital relative to output, notwithstanding large fluctuations in the enterprise value of the corporate sector. We find that it does, at least in the data since World War II. The reason is that the shocks to current and expected factorless income that are the main driver of changes in valuations do not generate large fluctuations in the expected return to capital, and thus in incentives for firms to change investment. In Figure 16 we show the model's implications for the gap between the expected return to investment in physical capital and a short-term riskless rate. We see that this expected excess return to investment in measured capital is fairly steady around 5 percentage points, with exceptions for the period in the early 1980's and in 2022 and in the period prior to World War II. In Figure 14 we show our model's implications for the level of the expected after-tax return to investment in measured capital. We see that our model accounting is consistent with the view that the expected returns to capital have been falling in recent decades. This reflects a combination of a rising depreciation rate for physical capital coupled with a declining share of income accruing to owners of capital. However, these two return-reducing factors are partially offset by declining corporate income tax rates.

Finally, in Section 8, we conclude. We provide a full description of the data we use as well as alternative graphs and our model of the pricing kernel in the Appendix.

2 Related Literature

A full reconciliation of macroeconomics and finance would achieve, at a minimum, three objectives:

1. a general equilibrium theory of the pricing kernel consistent with the asset prices confronting households as they make their consumption and savings decisions,
2. an accounting for the volatility of market valuations of U.S. corporations, and
3. an accounting for the observed investment/capital stock decisions of U.S. corporations given those asset prices.

There is a huge literature in macro-finance aimed at the first point. In particular, following Lucas (1978), this literature aims to develop models of the marginal utility of the marginal investor to resolve the equity premium puzzle of Mehra and Prescott (1985), manifest here as the high unconditional average rate of return observed on claims on the corporate sector

(2022) for an exposition of this point in a model economy with capital.

in both the public firm and IMA data. We do not attempt such an economic model of the marginal investor. We leave such modeling to future work.[4]

Instead, we build a slightly modified stochastic growth model to use as an accounting framework to address the second and third objectives above, taking a model of the pricing kernel as given. That is, we aim to simultaneously account for the observed fluctuations in the value of U.S. corporations year-by-year since 1929 and for measured U.S. corporate holdings of capital and investment in capital over this time period.

As discussed in Gomme, Ravikumar, and Rupert (2011) and as implied by the work of Tobin (1969), one of the challenges to accounting for the volatility of market valuations and returns of U.S. corporations together with the relatively smooth data on measured capital-output ratios and accounting returns to capital in a standard stochastic growth model is that in such a model the value of firms is always equal to the value of their installed capital. In our model, this link is broken by the introduction of factorless income and considerations of corporate taxation.

In contrast, much of the macro-finance literature aimed at the second and third objectives above has taken a different approach, relying on time-variation in the risk premium on investment in the capital stock or adjustment costs for that investment to reconcile the high volatility of corporate valuations and the relatively smooth evolution of the stock of measured capital. Jermann (1998), Gourio (2012), Ilut and Schneider (2014), Basu and Bundick (2017), Hall (2017), Cambell, Pflueger, and Viceira (2020), and Basu et al. (2023) are examples of stochastic growth models with time-varying risk premia on capital arising from a variety of different sources. See also Cochrane (1991), Merz and Yashiv (2007), Philippon (2009), and Jermann (2010).

We depart from this literature in accounting for the volatility of corporate valuations based on a model of fluctuations in expected cash flows to owners of firms rather than variation in discount rates.[5] To provide such an accounting, our model combines two key ingredients. First, we include a time-varying wedge between corporate revenue and costs associated with measured capital and labor that generates that we call *factorless income* following Karabarbounis and Neiman (2019). Second, we use a model of the impact of rates of return and taxes on the cash flow to owners of capital and the valuation of that cash flow

[4] Tallarini (2000) and Kaltenbrunner and Loechster (2010) are important papers showing how to reconcile standard business cycle fluctuations with a high equity premium in standard stochastic growth models with a representative agent with recursive preferences. These models have an advantage of being fairly tractable using standard approximation techniques. We conjecture that incorporation of shocks to factorless income in models such as these might be a fruitful avenue for developing fully equilibrium macrofinance models, albeit with constant risk premia over time.

[5] We pursue this theme further in a companion paper Atkeson, Heathcote, and Perri (2024) that models the volatility of stock prices based on fluctuations in expected cash flows using CRSP data on price per share and dividends per share.

based on the framework of Hall and Jorgenson (1967). We model taxes in a similar way to Gravelle (1994), Gravelle (2006), and Barro and Furman (2018). As in McGrattan and Prescott (2005) and McGrattan (2023) we find that taxes play an important role in shaping our model's implications for the valuation of and marginal returns to measured capital.

In our accounting of the data since World War II, the high volatility in the valuation of U.S. corporations is driven primarily by shifts in investors' expectations of the share of factorless income in corporate output in the long run. Investment in measured capital, on the other hand, is driven by more near-term considerations such as one-year interest rates, growth rates, corporate tax rates, depreciation rates, and changes in the relative price of capital goods. Thus, investment and measures of Tobin's Q are only weakly connected in our model (see Abel and Eberly (2012) for a related argument).[6] In that vein, our accounting for the stability of the ratio of physical capital to output in the face of falling risk free interest rates, changing tax rates, and rising depreciation rates in the past several decades is related to that in Gutiérrez and Philippon (2017), A declining share of income accruing to physical capital coupled with higher average depreciation have been important forces tending to depress investment. But these forces in our accounting have been offset by declining corporate tax rates and declining expected output growth, which in our valuation framework translates to lower required expected returns to all assets. We see this stable capital to output outcome as largely coincidental. In particular, We note that we do see large changes in the ratio of measured capital to corporate gross value added in the data prior to World War II. Moreover, these large changes in the capital-output ratio accounted for large changes in both the ratios of enterprise value and free cash flow to gross value added in that time period.

Our focus on shocks to current and future factorless income is closely related to the arguments of Lustig and Van Nieuwerburgh (2008) and Greenwald, Lettau, and Ludvigson (2023) that shocks to the distribution of income between workers and owners of firms have been an important driver of fluctuations in the valuation of U.S. corporations. Our principal contribution relative to these papers is to add consideration of physical capital and investment. We follow a large recent literature in macro-finance that builds on these ideas. See, for example, Caballero, Farhi, and Gourinchas (2017), Farhi and Gourio (2018), Crouzet and Eberly (2018), Philippon (2019), Eggertsson, Robbins, and Wold (2021), and Crouzet and Eberly (2023). With the notable exception of Crouzet and Eberly (2023), these papers do not account year-by-year for both corporate valuations and changes in capital investment over a long time period.

In building our accounting model, we make the stark assumption that the production

[6] We note that our findings about the volatility of cash flows to owners of U.S. corporations differ from the findings in Hall (2003). We believe that this is due to our use of free cash flow rather than EBITDA and to the time periods considered.

function relating measured capital and labor to aggregate output has remained stable over the past 100 years. It is this assumption that allows us, through the model, to measure the share of factorless income in corporate gross value added year-by-year from data on tax rates and the share of labor compensation in corporate gross value added. Given these estimates of the share of factorless income and our model of the production function, we can then compute the rental rate on measured capital, the share of rental income on measured capital in corporate gross value added, and the pre- and post-tax returns to physical capital implied by the data as interpreted through our model. We evaluate our model's fit to the data on investment and capital stocks based on a comparison of these model-based estimates of the post-tax returns to physical capital to risk free interest rates plus a constant risk premium on measured capital. That is, we ask whether our model's implications for the post-tax returns to measured capital are consistent with observed risk-free interest rates plus a constant risk premium.

In this regard, our work is closely related to recent work by Barkai (2020) and Karabarbounis and Neiman (2019) who both use a Hall and Jorgenson (1967) style measurement framework to estimate the rental rate on measured capital and the corresponding share of rental income on measured capital in gross value added. This prior work differs from ours in that it starts with data on risk free rates and an estimate of the risk premium on capital to estimate the rental rate on measured capital without imposing restrictions on the production function. If the specification of our model is correct (and measurement is without error) then the estimates of rental income on measured capital obtained from our framework and their framework should coincide. It appears that our results do roughly coincide for the period after the late 1980's. Prior to 1980 there is a discrepancy between our results and those of Karabarbounis and Neiman (2019) regarding returns to capital and the amount of factorless income. We aim to explore this discrepancy in greater detail in future work.

In the data, as noted by Gomme, Ravikumar, and Rupert (2015), Reis (2022), Harper and Retus (2022), and others, the accounting returns to capital in the corporate sector (measured by the ratio of Net Operating Surplus pre and post tax to installed capital) have remained remarkably constant since at least 1960, even as measures of the risk free interest rates have fallen quite sharply. Our accounting model is consistent with these accounting returns data. And yet we find a falling return to measured capital in recent decades, a fall of similar magnitude to the decline in risk free rates. In fact, we find that in recent years, the return to measured capital is falling close to the threshold for dynamic inefficiency of Abel et al. (1989). In our model, these differential trends between accounting returns and true returns to capital emerge because only a (declining) portion of Net Operating Surplus is compensation of measured capital, while the remaining (rising) portion contributes to factorless income.

Once we differentiate appropriately between free cash flow to physical capital and free cash flow associated with factorless income, we find that expected returns to all assets appear to declining over time, and at similar rates.

In our measurement, we have abstracted from the role of unmeasured intangible capital in accounting for fluctuations in the value of the U.S. corporate sector. Many papers consider the role of unmeasured intangible capital in driving the boom in the market valuation of U.S. firms in recent decades. See, for example, Hall (2001), McGrattan and Prescott (2010) and Crouzet et al. (2022). Eisfeldt and Papanikolaou (2014), Belo et al. (2022), Eisfeldt, Kim, and Papanikolaou (2022) and the papers cited therein argue that measured of intangible capital drawn from firms' accounting statements that is not included in the National Income and Product Accounts help account for the valuation of firms in the cross section. We see this as a fruitful avenue for future research, but we see two hurdles that should be overcome in developing this hypothesis.

First, the aggregate data cited in Corrado et al. (2022) are not favorable to the hypotheses that changes in the stock of unmeasured capital have contributed importantly to fluctuations in the value of the U.S. corporate sector because these data exhibit no trend in the stock of this unmeasured capital relative to value added over the past decade or more. Second, we suggest that a model of the variability of the market valuation of the U.S. corporate sector over the past century based on fluctuations in the stock of unmeasured capital held by U.S. corporations should also account for observed flows of free cash flow to owners of these corporations, as these cash flows are invariant to failure to measure investment; see Atkeson (2020).

We now turn to our discussion of the IMA data.

3 Measures of Corporate Value, Cash Flows, and Returns

In this paper, we focus on valuation and cash flow measures in the data from the Integrated Macroeconomic Accounts (IMA) closest to those concepts in a standard macroeconomic stochastic growth model. We consider a model in which firms are entirely equity financed, have no financial assets, and pay out all of their after-tax gross operating surplus less investment expenditures each period to firm owners. We refer to the concept of the value of the U.S. corporate sector that we use in this paper as *enterprise value*. We refer to the concept of cash flows for the U.S. corporate sector that we use in this paper as *free cash flow from operations*, or free cash flow for short.

We use the IMA data to construct a measure of enterprise value for the U.S. corporate sector as the sum of the market value of the equity and financial liabilities less the financial assets of U.S. corporations.[7]

Our measure of free cash flow in the IMA data is equal to after-tax gross operating surplus less investment expenditures of U.S. corporations. These valuation and cash flow measures are similar to those used in Hall (2001). Full details of our data construction for these and all other variables used in the paper are given in Appendix A.

We plot our valuation and cash flow measures relative to the gross value added of the U.S. corporate sector in Figure 1. We show enterprise value in the left panel in blue and free cash flow in the right panel in red. We see that both enterprise value and free cash flow are quite volatile relative to the gross value added of the U.S. corporate sector.

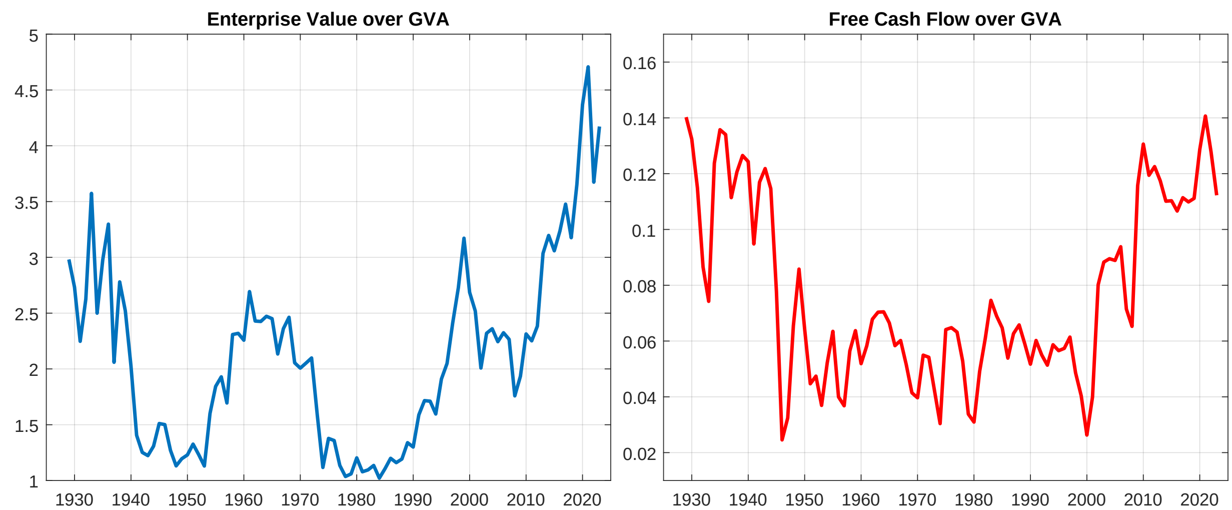

Figure 1: Left Panel: The Enterprise Value of U.S. Corporations over Corporate Gross Value Added. Right Panel: Free Cash Flow from U.S. Corporations over Corporate Gross Value Added. 1929-2023

In the left panel of Figure 2 we show the ratio of free cash flow to enterprise value. This valuation ratio shows considerable business cycle fluctuations but appears to be stable over the long term.

The right panel of this figure shows the ratio of enterprise value to gross value added in blue and a predicted value of this ratio if enterprise value were a fixed multiple of free cash flow in red.[8] We see in this panel that the low frequency fluctuations in the ratio of

[7]This measure of enterprise value for the Financial and Non-Financial corporate sectors is reported on Table B1 "The Derivation of U.S. Net Wealth" of the Financial Accounts of the United States. See https://www.federalreserve.gov/econresdata/notes/feds-notes/2015/us-net-wealth-in-the-financial-accounts-of-the-united-states-20151008.html

[8]We use a valuation multiple of free cash flow of $1/0.032 = 31.25$ in this calculation.

enterprise value to gross value added appear to be fairly well accounted for by low frequency fluctuations in the ratio of free cash flow to gross value added, when those are valued at a constant price dividend ratio.[9]

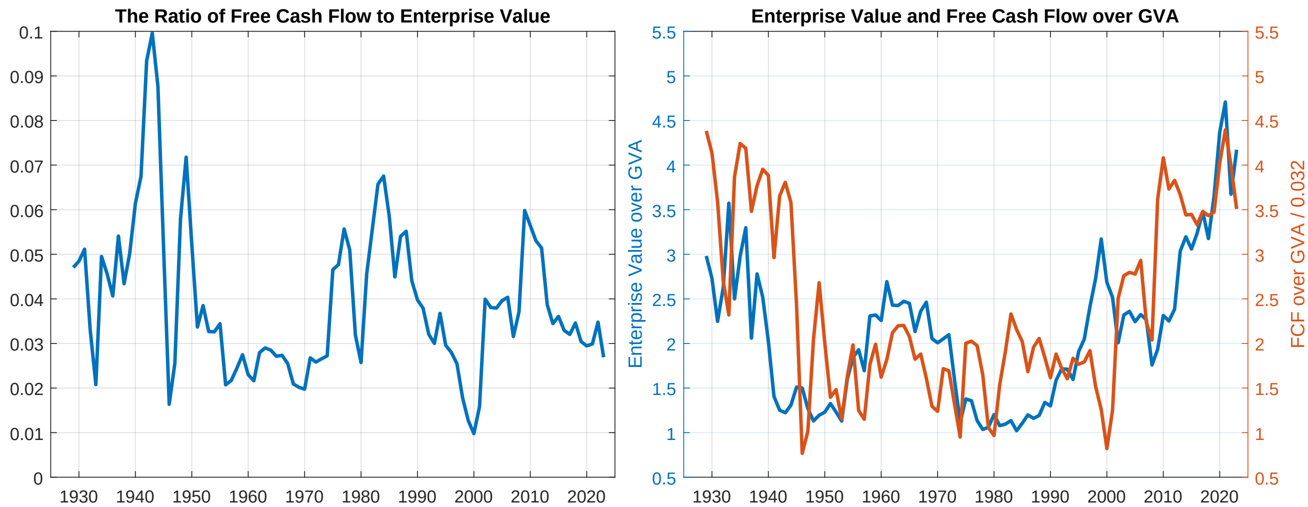

Figure 2: Left Panel: Free Cash Flow from U.S. Corporations over Enterprise Value. Right Panel: Enterprise Value (left axis) Actual and and Predicted from Corporate Free Cash Flow using a valuation multiple of $1/0.032 = 31.25$ (right axis) over Gross Value Added. 1929-2023

We also use the IMA to construct a market valuation of the equity of U.S. corporations (both publicly traded and closely held corporations) and a corresponding cash flow measure of monetary dividends paid to the owners of these corporations. We include these alternative measures of valuation and cash flow to facilitate comparisons between the IMA data and work using data from CRSP and Compustat for publicly traded firms. We show the IMA measures for the value of equity and for dividends relative to gross value added for the U.S. corporate sector in red in the left and right panels of Figure 3. Our measures of enterprise value and free cash flow are in blue.[10]

We see in the left panel of Figure 3 that the fluctuations in the market value of equity and enterprise value for U.S. corporations are tightly linked. By comparing the different scales for enterprise value (left axis) and equity (right axis), we see that enterprise value is consistently about 50 percentage points of gross value added larger than the market value of

[9]A simple variance decomposition of fluctuations in the log of the ratio of enterprise value is consistent with each of these components playing an important role. The variance of the log of the ratio of enterprise value to corporate GVA from 1929-2022 is 0.1476. The variance of the log of the ratio of free cash flow to corporate GVA is 0.1905. The variance of the log of the ratio of enterprise value to free cash flow is 0.1795. The covariance between these two series is -0.1112.

[10]We show an analogous plot for the enterprise value and the market value of publicly traded equities in Appendix Figure B.1.

equity. This difference between enterprise value and equity value corresponds to net debt of the U.S. corporate sector.

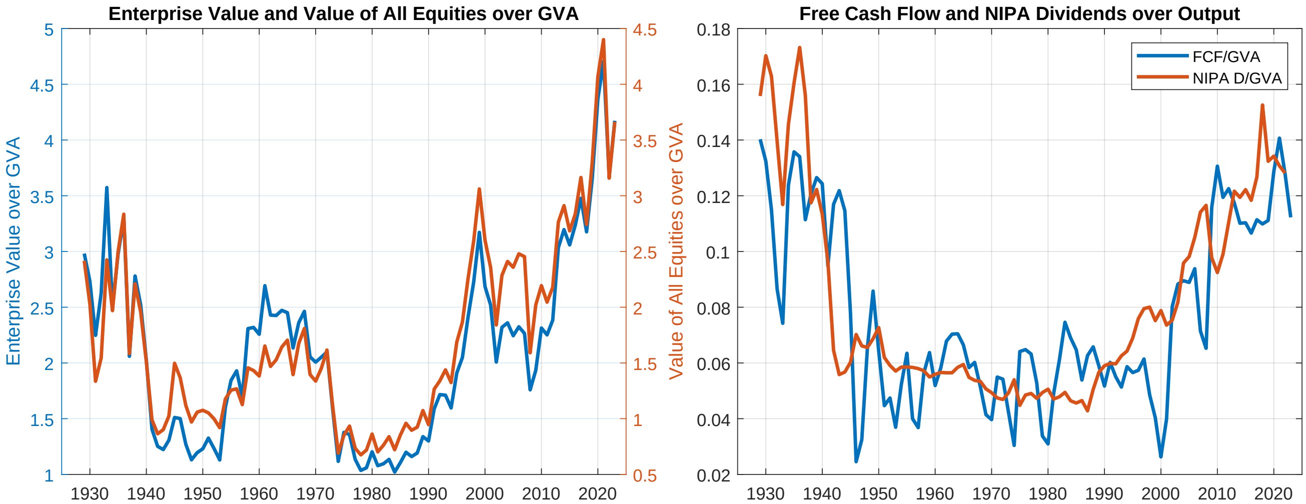

Figure 3: Left Panel: Enterprise Value (left axis) and Equity Value (right axis) of U.S. Corporations over Corporate Gross Value Added. Right Panel: Free Cash Flow and NIPA Monetary Dividends Paid over Corporate Gross Value Added. 1929-2023. NIPA Dividends Paid data are not yet available for 2023

We now consider properties of the annual returns on enterprise value and equity implied by these two sets of valuation and cash flow measures from the IMA.

We compute the returns on enterprise value from the perspective of a household in a stochastic growth model that owns the entire corporate sector and receives all cash paid out by that sector. Using that perspective, we denote enterprise value at the end of period t as V_t, free cash flow in period $t+1$ as FCF_{t+1}, and construct realized returns on enterprise value each year as

$$\exp(r^V_{t+1}) = \frac{FCF_{t+1} + V_{t+1}}{V_t}$$

We deflate these and all nominal returns by the growth in the PCE deflator to compute realized real returns.

We compute realized returns on equity from the perspective of a household that purchases equity at the end of period t at price V^E_t, collects dividend payments in year $t+1$, D^{IMA}_{t+1}, and sells that equity realizing a capital gain corresponding to the IMA reported revaluation of outstanding equity at $t+1$, $REVAL^E_{t+1}$. We compute this realized return as

$$\exp(r^E_{t+1}) = \frac{D^{IMA}_{t+1} + REVAL^E_{t+1} + V^E_t}{V^E_t}$$

This calculation of returns is closer to what is done in CRSP or Standard and Poors' data for public firms.

We report some basic statistics of the mean and standard deviations of log real returns using these two return concepts as well as analogous return and dividend growth statistics computed using CRSP returns on the value-weighted portfolio of NYSE, AMEX, and NASDAQ stocks in Table 1. We see in this table that all three measures of returns have similar means and standard deviations. We also see that the standard deviation of growth in log free cash flow is much higher than for the other two measures of dividends. This difference in volatility is readily visible in the right panel of Figure 3. We see in that figure that IMA Dividends smooth out the higher frequency fluctuations in free cash flow that are likely due to business cycle fluctuations in investment.

Table 1: Mean and Standard Deviation of Real Log Returns and Log Dividend Growth on Enterprise Value, IMA Equity, and CRSP Value-Weighted Portfolio

Return	Time Period	Mean Return	Std Return	Std D growth
Enterprise Value	1929-2023	0.073	0.146	0.280
IMA Equity	1929-2022	0.076	0.173	0.073
CRSP VW	1929-2023	0.062	0.193	0.138

In Figure 4 we examine the extent to which these measures of realized real returns on enterprise value and on IMA equity line up with measures of realized real returns computed using the CRSP value-weighted portfolio. In the left panel, we show a scatter plot of realized annual returns on the CRSP portfolio on the x-axis and returns on enterprise value on the y-axis. The red line is the 45 degree line. We show the corresponding scatter plot for CRSP returns and realized returns on IMA equity in the right panel. We see in the figure that both measures of returns constructed from the IMA data line up quite well with measures of equity returns from the CRSP database.[11]. The correlation of returns on enterprise value with those on the value-weighted CRSP portfolio is 0.943 for the period 1929-2023. The corresponding correlation for IMA equity returns with CRSP returns is 0.981 for 1929-2022.[12]

[11] We show the same scatter plots using data from the 1946-2022 time period in Appendix Figure B.2

[12] Note that one would expect some deviation of returns on enterprise value from returns on equity given the presence of net debt documented in the left panel of Figure 3.

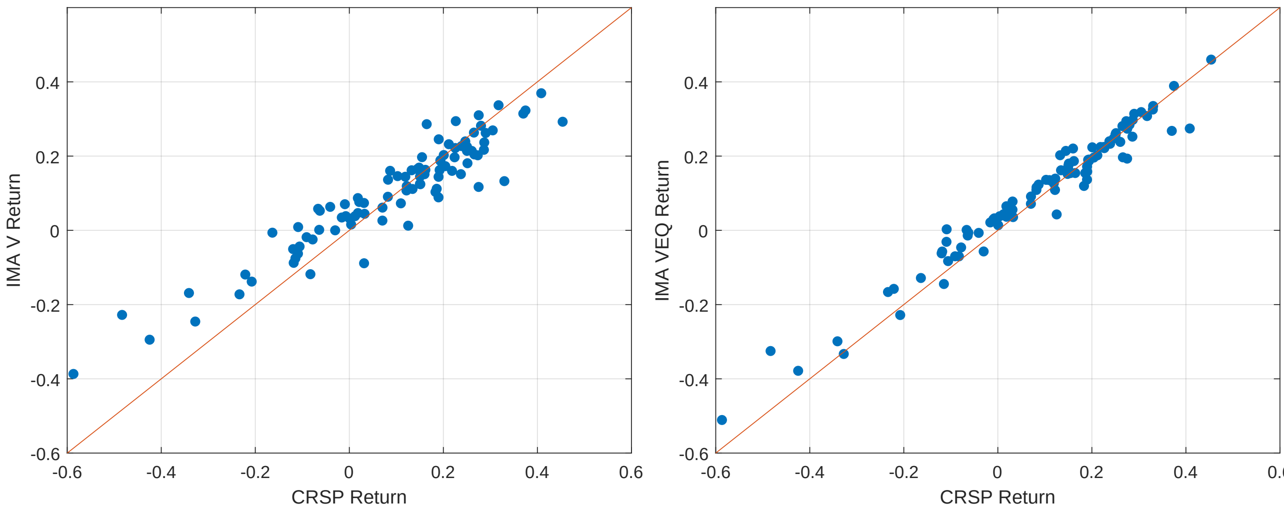

Figure 4: Left Panel: Realized Returns on Enterprise Value vs. CRSP Value-Weighted Return 1929-2023. Right Panel: Realized Returns on IMA Equity Value vs. CRSP Value-Weighted Return: 1929-2022

We now conduct one final comparison of our measures of free cash flow and enterprise value in the IMA data with analogous measures obtained from Compustat data on the financial accounting statements of publicly traded firms.[13] We expect to see differences in these measures of cash flows and valuation from these two data sets for many reasons, two of which stand out.

First, the IMA data are constructed to cover both publicly traded and closely held corporations, while the Compustat data cover only publicly traded corporations.[14] This conceptual distinction between the two data sets should act to make measures of free cash flow and enterprise value larger in the IMA data than corresponding estimates from Compustat data.

Second, the IMA data are constructed to cover only U.S. resident corporations. A U.S. resident corporation is an entity incorporated in the United States. Thus, these corporations include the U.S. subsidiaries of foreign multinational corporations and exclude the foreign subsidiaries of U.S. multinational corporations. In contrast, Compustat data covers the worldwide operations of a list of public companies that are determined to be U.S. corporations

[13]In the Compustat data, our measure of free cash flow is computed (following Adame et al. 2023) as Operating Activities–Net Cash Flow (OANCF) minus capital expenditures (CAPX). Enterprise value is computed as Total Market Value (MKVALT) plus Total Liabilities (LT) minus current assets total (ACT), which includes cash and other short term investments, receivables, inventories, and other current assets. Further details are given in Appendix A

[14]For a discussion of the methodology used in the IMA to value closely held corporate equities see https://www.federalreserve.gov/econresdata/notes/feds-notes/2016/corporate-equities-by-issuer-in-the-financial-accounts-of-the-united-states-20160329.html

in terms of the entity listing equity on U.S. markets. (See Atkeson, Heathcote, and Perri 2023 for further discussion of this point.) To the extent that the foreign subsidiaries of U.S. multinational corporations generate more free cash flow and contribute more to enterprise value than the U.S. subsidiaries of foreign multinationals, this conceptual distinction between these two data sets should act to make measures of free cash flow and enterprise value smaller in the IMA data than corresponding estimates from Compustat data.

In Figure 5 we plot free cash flow (left panel) and enterprise value (right panel) in the IMA and in Compustat, both divided by same denominator, which is gross value added of the corporate sector from the IMA. The left panel shows that in both Compustat and the IMA data the share of free cash flow in GVA roughly doubles from the early 1990s to the late 2000s. The right panel shows that enterprise value relative to IMA GVA increases by a similar amount in both data sets. Figure 6 shows that the ratio of free cash flow to enterprise value therefore appears relatively stable over time around a similar level in both the IMA and the Compustat data.

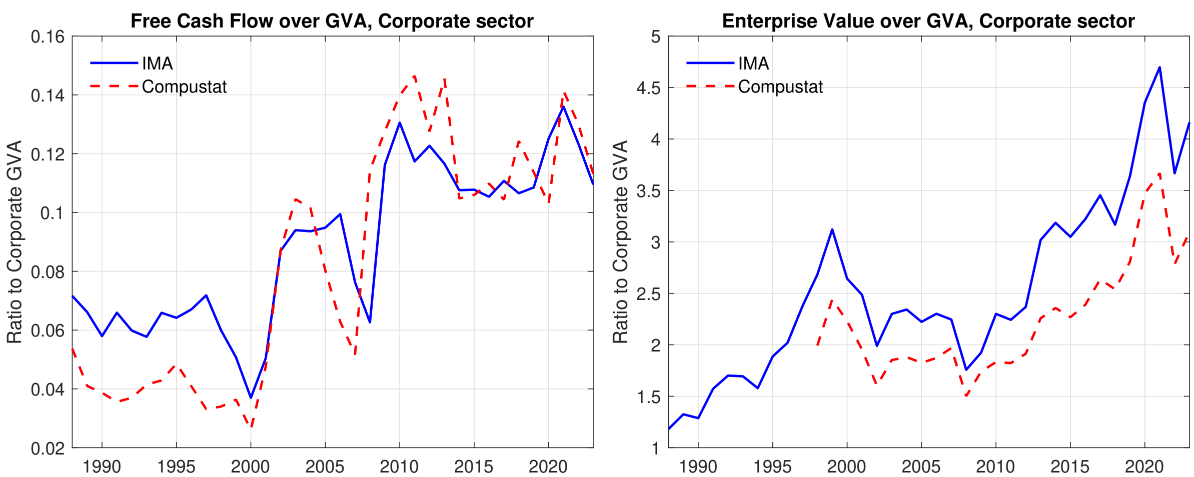

Figure 5: Free Cash Flow and Enterprise Value in the IMA and in Compustat

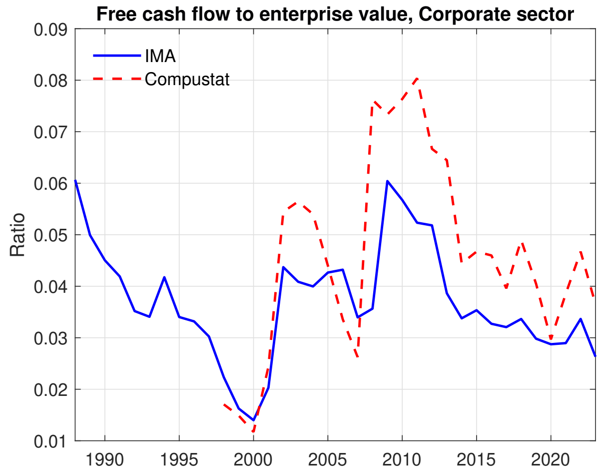

Figure 6: Ratio of Free Cash Flow to Enterprise Value in the IMA and in Compustat

The close correspondence between measures of value and returns for claims on the U.S. corporate sector from the Integrated Macroeconomic Accounts with measures of value and returns constructed from CRSP and Compustat data on public firms gives us confidence that the Integrated Macroeconomic Accounts are a useful data source for further work in macrofinance aimed at offering an integrated account of aggregate corporate valuations and cash flows.

We start on that agenda in the remainder of this paper.

4 Revisiting Campbell-Shiller Regressions

In Figure 1 we provide evidence that the ratio of U.S. corporate free cash flow to owners of firms has fluctuated a great deal relative to gross value added of the U.S. corporate sector over the period 1929-2023. The right panel of Figure 2 provides suggestive evidence that these fluctuations in free cash flow relative to GVA might account for a substantial portion of observed fluctuations in the enterprise value of U.S. corporations relative to the output of the U.S. corporate sector over this time period. But it is also clear from the left panel of that figure that the ratio of free cash flow to enterprise value (a dividend price ratio) is not constant over this time period so that fluctuations in this valuation ratio also play a significant role in accounting for fluctuations in the ratio of enterprise value to corporate sector output. Thus, to provide a full account of the drivers of fluctuations in the ratio of enterprise value to corporate value added, we must also provide an account of the drivers of fluctuations in the ratio of free cash flow to enterprise value, or, in other words, of the fluctuations over time in the dividend-price ratio for enterprise value.

Conceptually, a dividend price ratio can move because agents beliefs regarding the growth of future cash flows from that asset have changed or because the expected returns that agents demand to hold the asset have changed. Campbell-Shiller regressions are one widely-used methodology for decomposing fluctuations in dividend-price ratios into a portion due to fluctuations in expected future rates of return and a component due to fluctuations in the expected growth of future cash flows to owners of firms. The goal of these regressions is to determine the extent to which movements in dividend-price ratios forecast changes in future growth of cash flows or changes in future realized returns.

Prominent papers in this literature run these regressions with data on publicly traded firms and argue that fluctuations in expected growth of future cash flows to owners of firms account for at best a small portion of the observed fluctuations in dividend-price ratios for public equity because movements in these dividend price ratios show little ability to forecast future growth of cash flows.[15] We discuss these findings in detail in a companion paper to this one: Atkeson, Heathcote, and Perri (2024).

In this section, we revisit the results of Campbell-Shiller regressions using our data on enterprise value and free cash flow for the U.S. corporate sector. In this regard, we follow prior work by Larraine and Yogo (2008) who argued using similar data that fluctuations in valuation ratios do have considerable ability to forecast future cash flow growth when using valuation and total cash flow measures drawn from the *Financial Accounts of the United States*. We find similar results here using our concepts of enterprise value and free cash flow.

We also follow prior work by Greenwald, Lettau, and Ludvigson (2023) who emphasize the role of shocks to the share of value added flowing to owners of firms in driving changes in ratios of corporate valuation to output. We conduct separate Campbell-Shiller regressions examining the extent to which movements in free cash flow to enterprise value ratios forecast changes in future growth of corporate gross value added or growth in the share of free cash flow in corporate gross value added. We find that the substantial majority of cash flow predictability is in the share of free cash flow in corporate gross value added.

The regressions we consider are derived following the analysis in Cochrane (2011) leading to his Table II. We spell this logic out in detail for readers who may not be familiar with this standard material in finance.

The realized return on any asset with positive dividends can be written as

$$\exp(r_{t+1}) = \frac{P_{t+1} + D_{t+1}}{P_t} = \left[\frac{\frac{P_{t+1}}{D_{t+1}} + 1}{\frac{P_t}{D_t}}\right] \frac{D_{t+1}}{D_t}.$$

[15]See Koijen and Van Nieuwerburgh (2011) and Campbell (2018) Chapters 5.3 - 5.5 for a textbook summary of this methodology and discussion of it in the literature.

A loglinear approximation to this equation gives realized log returns as

$$\hat{r}_{t+1} \approx -\rho\widehat{dp}_{t+1} + \widehat{dp}_t + \hat{g}_{Dt+1}$$

with $\widehat{dp}_t$ denoting the log deviation of the dividend-price ratio from its value at the point of approximation, $\hat{g}_{Dt+1}$ denoting the log deviation of the growth rate of dividends from its value at the point of approximation, and ρ being a constant of approximation determined by the price dividend ratio at the point of approximation given by

$$\rho \equiv \frac{\overline{P/D}}{\overline{P/D}+1}.$$

We have suppressed reference to the constant term in this approximation.

Rearranging terms relates the log dividend-price ratio to future realized returns, realized divided growth, and the future realized dividend-price ratio

$$\widehat{dp}_t \approx \hat{r}_{t+1} - \hat{g}_{Dt+1} + \rho\widehat{dp}_{t+1}.$$

If we iterate on this formula k times, we get a formula relating the current log dividend-price ratio to cumulative realized returns and dividend growth over horizon k and the terminal dividend-price ratio at that horizon

$$\widehat{dp}_t \approx \sum_{j=1}^{k} \rho^{j-1}\hat{r}_{t+j} - \sum_{j=1}^{k} \rho^{j-1}\hat{g}_{Dt+j} + \rho^k\widehat{dp}_{t+k}.$$

This formula holds for all realizations, so it holds in expectation as well. Thus, a movement in the dividend-price ratio for this asset at time t should correspond to a linear combination of movements in expected future returns, expected future dividend growth, and the expected future dividend price ratio over a horizon of k future periods. If the dividend price ratio is stationary over time, the final term in this expression should go to zero as k gets large since $\rho < 1$ by construction. This argument leads to the standard decomposition of changes in current dividend price ratios into changes in subsequent returns and dividend growth.

We now run three regressions based on this approximation formula given by

$$\sum_{j=1}^{k} \rho^{j-1}\hat{r}_{t+j} = \alpha_r^k + \beta_r^k\widehat{dp}_t + \epsilon_{r,t+k}$$

$$\sum_{j=1}^{k} \rho^{j-1} \hat{g}_{Dt+j} = \alpha_{gD}^{k} + \beta_{gD}^{k} \widehat{dp}_t + \epsilon_{gD,t+k}$$

$$\rho^k \widehat{dp}_{t+k} = \alpha_{dp}^{k} + \beta_{dp}^{k} \widehat{dp}_t + \epsilon_{dp,t+k}$$

Observe that if one imposes the log return approximation, the slope coefficients in these regressions satisfy the constraint

$$\beta_r^k - \beta_{gD}^k + \beta_{dp}^k = 1$$

That is, one can interpret these slope coefficients as indicative of the extent to which fluctuations in log-dividend price ratios are accounted for by fluctuations in expected returns, expected dividend growth, and expected future dividend price ratios.

We revisit results from running these regressions over the 1929-2023 time period using annual data on real returns from our data on enterprise value returns and real growth of free cash flow. Following the presentation of results in Cochrane (2011) Table II, in our Table 2, we first report estimated slope coefficients corresponding to future returns (β_r^k), future dividend growth (β_{gD}^k), and the future price dividend ratio (β_{dp}^k) using a regression in which we construct the cumulated returns and dividend growth terms on the right side of the first two regressions directly in the data at a fifteen year horizon, and then run the three regressions. We then present results at a horizon of 15 years implied by iterating on one-year ahead forecasts treating the three regressions as a VAR.

Table 2: Campbell-Shiller 15-year Horizon Regression Coefficients. IMA Free Cash Flow and Enterprise Value Data, 1929-2023.

	return β_r^k	dividend growth β_{gD}^k	future dp ratio β_{dp}^k
Direct $k = 15$	0.46	-0.71	-0.17
VAR $k = 15$	0.28	-0.71	0.01

We see in these regression results confirmation of the hypothesis that movements in the log ratio of free cash flow to enterprise value are largely a forecast of movements in subsequent real growth of free cash flow.

We note that it is possible to decompose the predictability of growth in free cash flow shown in this table into a component due to predictability of growth in the log of corporate GVA and growth in the log of the ratio of free cash flow to GVA. Running these regressions separately reveals that all of the predictability in the growth of free cash flow comes from the predictability in its share in GVA (this coefficient estimate is -0.74 in the direct regression and -0.68 in VAR-based estimate) rather than in the growth of corporate GVA.

We take away from these results that the conventional wisdom that fluctuations in expected future cash flows are not an important driver of fluctuations in dividend price ratios and that these valuation ratios are driven primarily, or even exclusively, by fluctuations in expected returns is worth reconsidering in light of these IMA data. We begin that reconsideration in the next section with a model that we use for accounting the the valuation of U.S. corporations, their cash flows, and their choices of investment in physical capital.

5 Model

We now introduce the model we use to account for these data.

Our model has two main components.

The first is a model of production and income in the corporate sector that is based on the standard stochastic growth model. The main modification we make to the standard model is that we assume that firms charge prices that include a time-varying wedge relative to cost, so that a portion of value added corresponds to a pure rent paid to the owners of firms. Following Karabarbounis and Neiman (2019), we refer to this income as *factorless income*. We note that this factorless income can be positive or negative. To the extant that firms have power to charge a markup over the costs of labor and measured capital, factorless income is positive. To the extent that managers of firms fail to earn surplus sufficient to cover the opportunity cost of the measured capital that they own, factorless income is negative. We also explicitly model corporate taxation, and how it impacts, cash flow, returns, and valuations.

The second component of our model is the pricing kernel M_{t+1}. The pricing kernel is used to value the stream of cash flow that firm owners receive, and to rationalize the choices of investment in physical capital observed in the data through the capital Euler equation.

Our analysis of the model will proceed in steps. First, we show that with a minimal set of assumptions, we can use the model structure and the Integrated Macroeconomic Accounts to decompose the series for corporate free cash flow described earlier into a portion of cash flow accruing to owners of measured capital, and a portion accruing to owners of claims to factorless income.

Second, we show how the framework can be used to decompose aggregate enterprise value into the market value of cash flows to the owners of the measured capital stock and the market value of claims to factorless income. We show that in the data prior to World War II, much of the fluctuations in enterprise value relative to corporate output correspond fluctuations in the stock of measured capital relative to corporate output, while after World War II, fluctuations in the market value of measured capital account for little of the large observed swings in enterprise value.

Third, we formulate a specific stochastic process for factorless income in the economy, and impose some restrictions on our pricing kernel which determine how the risk of this cash flow is priced. Given these auxiliary assumptions we are able interpret fluctuations in the value of claims to factorless income as reflecting shocks to expected future factorless income.

Finally, fourth, we examine our model's implications through the capital Euler equation for the realized and expected excess returns to investment in measured capital over and above a short term risk free rate.

5.1 Technology

We start by describing the production technology.

Aggregate output, corresponding to gross value added of the corporate sector, GVA_t, is given by a Cobb-Douglas production function

$$GVA_t = K_t^{\alpha}(Z_t L)^{1-\alpha}, \tag{1}$$

where K_t is the stock of physical capital in units of capital services, L is labor, which is inelastically supplied, and Z_t is a shock to aggregate productivity. We will assume that the share of capital in production, denoted by α, is constant over time.

The evolution of the stock of capital services is given by

$$K_{t+1} = (1-\delta_t)K_t + I_t,$$

where δ_t is a time-varying physical depreciation rate for capital services and I_t is investment in new capital services. Note that we assume here that there are no investment adjustment costs.

The terms K_t and I_t are not directly measured in the data. Instead, the IMA report end of period nominal values for the stock of capital at replacement cost, nominal investment expenditures, nominal consumption of fixed capital, and nominal revaluations of the stock of capital carried into the period due to changes in the replacement cost of that capital. In our model, we use P_t to denote the nominal price level and Q_t to denote the real price of capital goods. We write the real end-of-period t replacement cost of capital as $Q_t K_{t+1}$, real investment expenditure in period t as $Q_t I_t$, real consumption of fixed capital as $\delta_t Q_t K_t$, and real revaluations of the replacement value of capital carried into period t as $(Q_t - Q_{t-1})K_t$.

Thus, the accounting in the IMA data is

$$\underbrace{Q_t K_{t+1}}_{ReplacementCost_{t+1}} = \underbrace{Q_{t-1} K_t}_{ReplacementCost_t} + \underbrace{(Q_t - Q_{t-1}) K_t}_{Reval_t + Other_t} - \underbrace{\delta_t Q_t K_t}_{CFC_t} + \underbrace{Q_t I_t}_{Investment_t}$$

The growth rate for Q_t can be directly inferred by dividing the reported revaluation value by the reported end of t replacement cost.[16] One can then construct a series for $Q_t K_t$ and from that infer a series for δ_t given reported consumption of fixed capital.

For the purposes of interpreting valuations, it is helpful to conceptualize two types of firms operating in the economy. One type, which we call investment firms, holds measured capital, makes investment decisions, and earns income by renting out this capital at a rental rate per unit of capital services R_t^K. The second type of firm, which we call factorless income firms, rent capital and labor, whose wage rate is W_t, and use these inputs to produce the final good according to equation (1). These factorless income firms earn this income by selling output with a wedge μ_t between revenue and the cost of measured capital and labor in production. As discussed above, this wedge can be greater or less than one. We assume that both types of firms are 100 percent equity financed, and that both pay out all the free cash flow they generate as model dividends. We also assume that both types of firms seek to maximize the present value of model dividends payable to shareholders, where these dividends at date $t+k$ are discounted back to date t according to a common pricing kernel, $M_{t,t+k}$.[17]

5.2 Corporate Taxation

To construct measures of free cash flow for these firms we need to specify how they are taxed. We model two sorts of corporate taxes. First, we assume factorless income firms pay a proportional tax at a time-varying rate τ_t^s that applies to GVA_t. This tax in the model corresponds to indirect businss taxes in the data. Thus, we estimate τ_t^s by dividing the sum of "taxes on production and imports less subsidies" plus "business current transfer payments" from NIPA Table 1.14 by corporate gross value-added.

Second, we model corporate income taxes as follows, building on Gravelle (1994) and Barro and Furman (2018). We assume corporate income is taxed at a proportional rate τ_t^c. We assume that investment firms can fully expense economic depreciation and can also expense a constant fraction λ of net new investment. Given these assumptions, the effective tax rate on capital is approximately equal to $\tau_t^c(1-\lambda)/(1-\lambda\tau_t^c)$.[18] Factorless income firms pay the corporate income tax on their factorless income, but are entitled to a time-varying

[16]We work with nominal data, so this procedure identifies the growth of the nominal price of capital goods.

[17]These discount rates are state- as well as date-specific.

[18]See Appendix C for the derivation and for an exact expression.

lump-sum tax credit, T_t^L. We use this credit to reconcile marginal tax rates with total taxes paid.

Given this model, total corporate income taxes paid are given by

$$Taxes_t^c = \tau_t^c \left[(1 - \tau_t^s) GVA_t - W_t L - \delta_t Q_t K_t - \lambda Q_t \left(K_{t+1} - K_t \right) \right] - T_t^L \tag{2}$$

We set the value for τ_t^c in each year t equal to corresponding value for the top rate of federal corporate income tax. We set $\lambda = 0.2$. These choices imply a time path for the effective tax rate on capital income similar to the one estimated by Gravelle (2006). Given those choices, we set the time path for T_t^L so that implied total corporate income tax revenue (equation 2) matches the series for "taxes on corporate income" in NIPA Table 1.14. Figure 7 plots corporate income tax revenue as a share of gross value added, and the time paths for the statutory and effective tax rates.

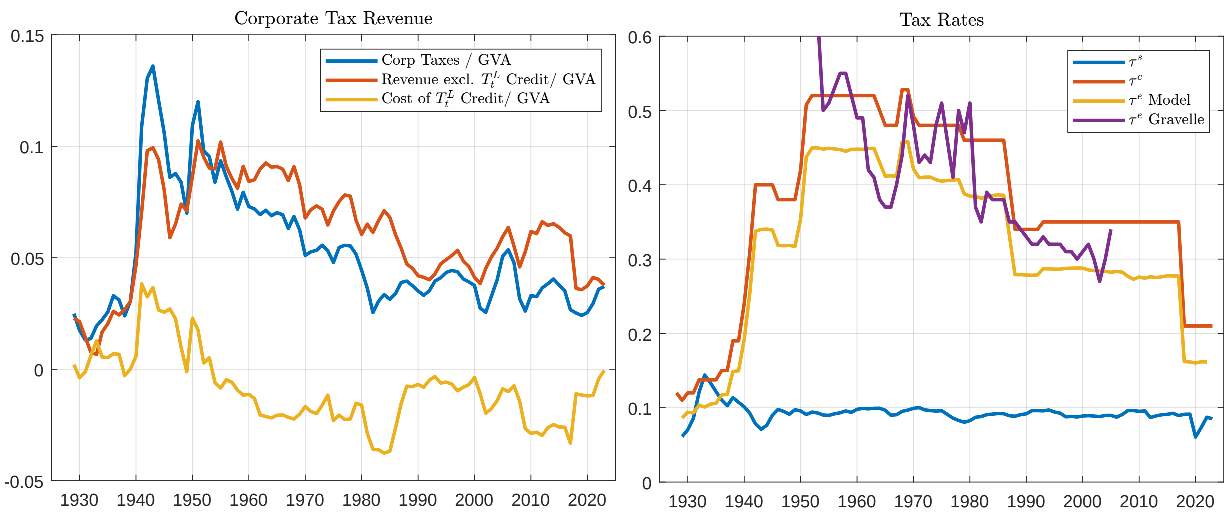

Figure 7: Left Panel: Corporate Income Tax Revenue 1929-2023. Right Panel: Model Tax Rates 1929-2023. τ_t^s is the value-added tax rate, τ_t^c is the statutory corporate income tax rate, and τ_t^e is the marginal effective tax rate on capital.

5.3 Drivers of Free Cash Flow

Given our assumed tax structure, total corporate free cash flow is

$$FCF_t = (1 - \tau_t^s) GVA_t - W_t L - Q_t I_t - Taxes_t^c. \tag{3}$$

Measuring total free cash flow for the corporate sector using equation (3) is straightforward given the IMA series for gross-value added, compensation, investment, and corporate taxes.

Figure 8 documents the contributions of these different components in accounting for observed changes in total free cash flow. To make the plot easier to read, we measure deviations of each component from their sample average, and filter the series to remove high frequency fluctuations. The message of this figure is that changes in labor's share of income, changes in investment, and changes in corporate taxes are all important drivers of total corporate free cash flow. Note, first, that the well-documented decline in labor's share of value-added in the post 2000 period accounts for essentially all of the observed increase in free cash flow over that period. At the start of the sample period, the key reason cash flow declines is that corporate taxes paid rise. Between the end of World War II and 2000, labor's share of income is relatively stable. Investment rises steadily as a share of value added, reducing free cash flow, but this trend is largely offset by a declining corporate tax burden.

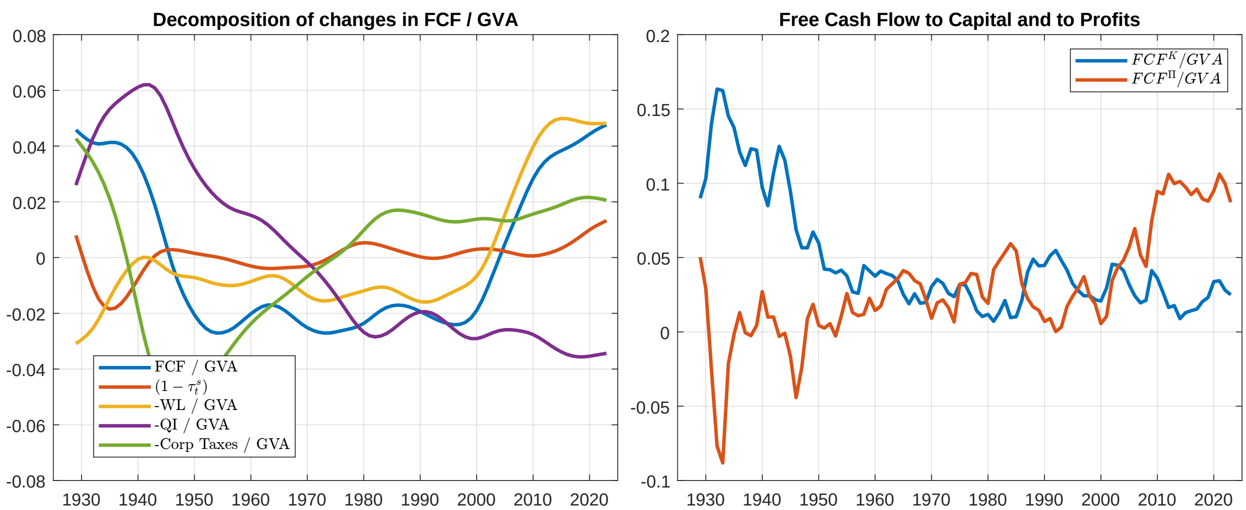

Figure 8: Left Panel: Decomposition of Total Corporate Free Cash Flow. For each component, we plot the Hodrick-Prescott trend value (smoothing parameter = 100) for the difference of the component from its sample mean. Right Panel: Decomposition of Free Cash Flow into Free Cash Flow to Capital and Free Cash Flow to Factorless Income Firms.

5.4 Income Shares

We use our model to split this free cash flow into a component going to owners of measured capital, and income to owners of claims to factorless income. After-tax free cash flows from investment-producing and factorless income firms are given, respectively, by

$$FCF_t^K = R_t^K K_t - Q_t I_t - \tau_t^c \left[R_t^K K_t - \delta_t Q_t K_t - \lambda Q_t \left(K_{t+1} - K_t \right) \right]$$

and

$$FCF_t^{\Pi} = (1-\tau_t^c)\Pi_t + T_t^L,$$

where

$$\Pi_t = (1-\tau_t^s)GVA_t - W_t L - R_t^K K_t \tag{4}$$

denotes pre-corporate-tax factorless income.

To construct these free cash flow series requires an estimate of capital rental income $R_t^K K_t$ or equivalently for Π_t. We now use our model structure to construct such a series.

In our model, factorless income firms solve static problems, choosing how much capital and labor to rent each period to minimize costs. Given the Cobb-Douglas production function, the optimal ratio of capital relative to labor services is given by

$$\frac{K_t}{L} = \frac{\alpha}{(1-\alpha)}\frac{W_t}{R_t^K} \tag{5}$$

These firms set prices net of value-added tax with a time-varying wedge μ_t over unit cost:

$$(1-\tau_t^s)GVA_t = \mu_t(W_t L + R_t^K K_t) \tag{6}$$

In Atkeson, Heathcote, and Perri (2023) we show how such wedges can be micro-founded as arising from Bertrand competition between more and less productive potential producers.

Given, equations (5) and (6), our model's implications for the division of gross value added into income shares is as follows. Share τ_t^s, accrues to the government as taxes on production and imports less subsidies. The remainder is divided according to

$$\frac{W_t L}{GVA_t} = (1-\tau_t^s)(1-\alpha)\frac{1}{\mu_t} \tag{7}$$

$$\frac{R_t^K K_t}{GVA_t} = (1-\tau_t^s)\alpha\frac{1}{\mu_t} \tag{8}$$

$$\frac{\Pi_t}{GVA_t} = (1-\tau_t^s)\frac{(\mu_t-1)}{\mu_t} \tag{9}$$

where $\frac{W_t L}{GVA_t}$ is the model equivalent of compensation of employees and the sum $\frac{R_t^K K_t}{GVA_t} + \frac{\Pi_t}{GVA_t}$ is the model equivalent of Gross Operating Surplus. Note that the model equivalent of Taxes on Corporate Income and Wealth is given as in equation 2 is not considered as an income share in NIPA.

Let κ_t denote free cash flow to owners of factorless income firms at date t, relative to gross-value added:

$$\begin{aligned}\kappa_t &\equiv \frac{FCF_t^{\Pi}}{GVA_t} = (1-\tau_t^c)\frac{\Pi_t}{GVA_t} + \frac{T_t^L}{GVA_t} \\ &= (1-\tau_t^c)(1-\tau_t^s)\frac{\mu_t - 1}{\mu_t} + \tau_t^L\end{aligned}$$

where $\tau_t^L = \frac{T_t^L}{GVA_t}$.

Equation (7) implies a tight link between fluctuations in the price-cost wedge μ_t and fluctuations in labor's share of income. Using that relationship one can express free cash flow to factorless income firms κ_t as a function of tax rates and labor's share:

$$\kappa_t = (1-\tau_t^s)(1-\tau_t^c) + \tau_t^L - \frac{(1-\tau_t^c)}{(1-\alpha)}\frac{W_t L_t}{GVA_t} \tag{10}$$

Thus, in this model, given tax parameters and a choice for the share parameter α, the path for factorless income as a share of corporate GVA can be identified given a path for compensation to labor as a share of gross value added, which we take straight from the IMA. Of course, given total free cash flow, and free cash flow to factorless income firms, we also have free cash flow to owners of capital.

The right panel of Figure 8 plots factorless income (κ_t) and free cash flow to capital as ratios to corporate GVA, given a choice of $\alpha = 0.29$. We will discuss the logic for this parameter choice below.

The ratio of factorless income to GVA is quite volatile. In addition, it generally appears to trend upward over time, and has risen sharply since 2000, from a share near zero, to around 10 percent of corporate gross value added. Mechanically, this is largely driven by the decline in labor's share of income over this period: see equation (10) and the left panel of the figure.

Free cash flow to capital as a share of gross-value added declines quite dramatically in the early decades of our sample, but appears relatively stable from around 1970 onward. Cash flow to capital was high during the Great Depression because investment and corporate taxes were both very low. Over time, the main driving of declining cash flow to capital has been rising investment as a share of gross value-added (see Figure 9). This in turn reflects an upward trend in the depreciation rate δ. Rental income from capital $R_t^K K_t$ in the model moves in lockstep with compensation of employees (see equations 7 and 8), so the declining labor share post 2000 also worked to reduce cash flow to capital.

Note that the path for free cash flow to capital that we find reflects a minimum of model structure: our assumption of a Cobb-Douglas production function with a constant capital share in costs α. As for factorless income, alternative values of α simply shift this measure of series up or down without changing the trend. We discuss how the measured decline in free cash flow to capital impacts our measurement of the returns to investment in physical capital in Section 7.

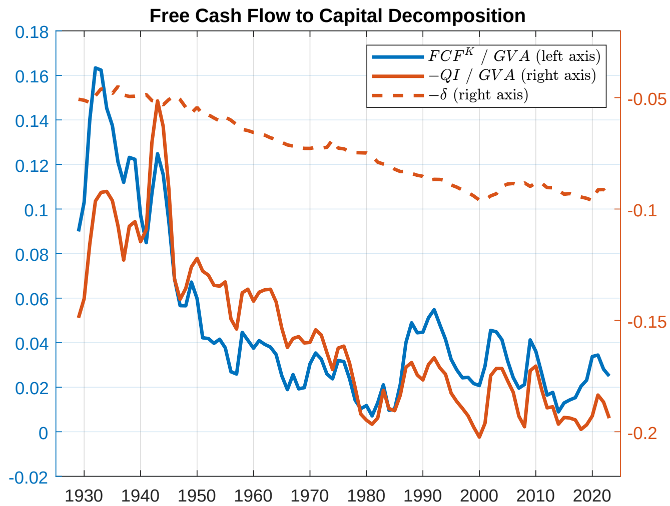

Figure 9: Free Cash Flow to Capital and the Contribution of Investment.

5.5 Firm Valuation

Enterprise value in our model is the expected discounted present value of free cash flow to owners of firms, with those present values computed using the model's pricing kernel M_{t+1}. Given our division of free cash flow FCF_t into a component that is factorless income $FCF_t^{\Pi} = \kappa_t GVA_t$ and a component that is free cash flow to capital FCF_t^K, it is natural to decompose enterprise value, denoted by V_t as the sum of the values of these two cash flows

$$V_t = V_t^K + V_t^{\Pi} \tag{11}$$

where V_t^K denotes the value of future free cash flow to capital and V_t^{Π} denotes the value of future factorless income.

The firm that owns and manages the physical capital stock takes as given an initial capital stock K_t and chooses future capital $\{K_{t+k}\}$ and after-tax free cash flow payable to owners $\{FCF_{t+k}^K\}$ for $k \geq 1$ to maximize

$$FCF_t^K + V_t^K$$

where

$$V_t^K = \sum_{k=1}^{\infty} \mathbb{E}_t \left[M_{t,t+k} FCF_{t+k}^K \right]$$

The first-order condition with respect to K_{t+1} is

$$\mathbb{E}_t\left[M_{t,t+1}\left[(1-\tau^c_{t+1})\left(R^K_{t+1}-Q_{t+1}\delta_{t+1}\right)+(1-\lambda\tau^c_{t+1})Q_{t+1}\right]\right]=(1-\lambda\tau^c_t)Q_t. \tag{12}$$

If investment firms choose investment according to the capital Euler equation (12) at every date, then one can show that the value of future free cash flow to capital is given by

$$V^K_t=(1-\lambda\tau^c_t)Q_tK_{t+1} \tag{13}$$

This result is independent of the specification for the pricing kernel $M_{t,t+k}$, but it does rely on the assumption that the production function is constant returns to scale, and that there are no investment adjustment costs.[19] Given this result, we measure the value of factorless income using the IMA data using the difference between enterprise value and the value of the claims to capital:

$$V^\Pi_t=V_t-(1-\lambda\tau^c_t)Q_tK_{t+1}. \tag{14}$$

Note that if $\lambda=1$ (full expensing of net investment) then a constant corporate tax rate does not distort investment: all the tax terms in equation (12) cancel out. However, firm value is depressed relative to the replacement cost of capital by a factor $(1-\tau^c_t)$. The intuition is that the capital tax depresses income to capital – and thus the market value of capital – which reduces the incentive to invest. But full expensing allowance sufficiently subsidizes the cost of new investment to exactly offset that effect.

Conversely, if $\lambda_t=0$, then investment and capital are depressed when $\tau^c_t>0$, but the value of the firm is equal to the replacement cost of its capital. See McGrattan and Prescott (2005) for a related discussion.

We show the breakdown of enterprise value relative to gross value added into these two components in Figure 10. The data in this figure are the key valuation facts that we wish to account for and to reconcile with the dynamics of cash flows described above.

In the left panel of this figure, we show enterprise value (in blue) and the market value of the capital stock (in red). In the right panel of this figure, we again show enterprise value (in blue) and the value of claims to factorless income (in red). We see in the left panel of this figure that between 1929 and World War II (WWII), fluctuations in the value of capital account for much of the fluctuations in enterprise value, but that after WWII, the ratio of the value of capital to value added has remained remarkably stable. In some periods, the market value of capital exceeds enterprise value. If one were to instead measure capital at replacement cost, the capital value would be higher still.

[19] We provide a proof of this result in Appendix D.

In the right panel of Figure 10, we see that it is fluctuations in the value of claims to factorless income that account for the majority of fluctuations in enterprise value after World War II. Note that in some periods the value of claims to factorless income is negative. Our valuation model will need to account for occasionally negative valuations for claims to factorless income even though actual such income is generally positive (Figure 8).

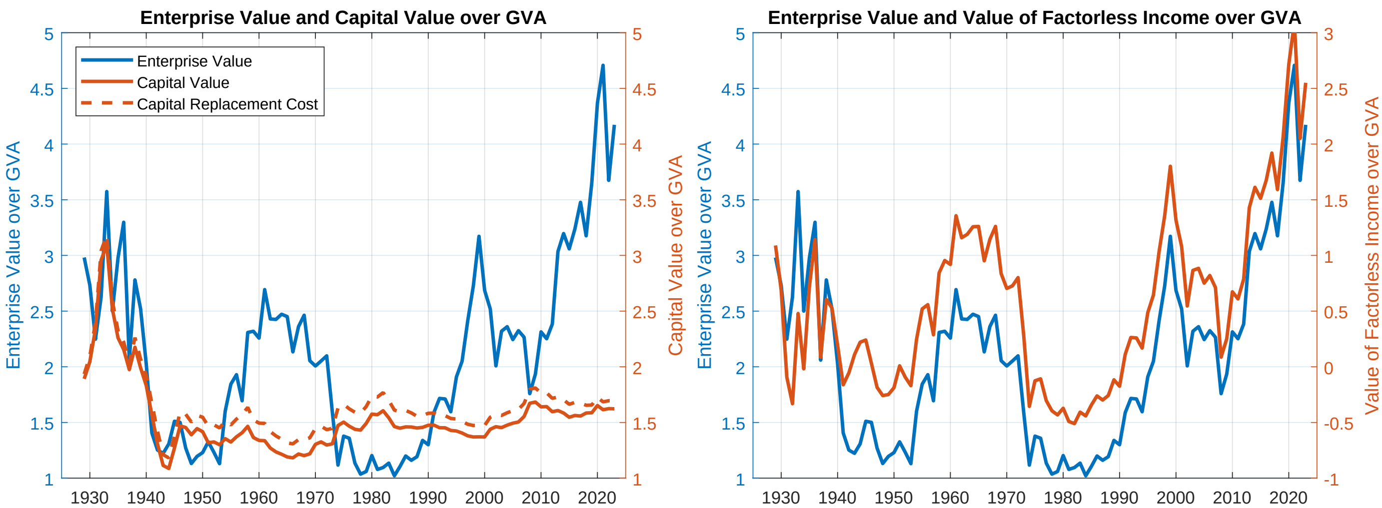

Figure 10: Left Panel: Enterprise Value (left axis) and Value of Capital Stock (right axis) over U.S. Corporate Gross Value Added. 1929-2023 Right Panel: Enterprise Value (left axis) and Value of Factorless Income (right axis) over U.S. Corporate Gross Value Added. 1929-2023

6 Valuing Factorless Income

We next turn to our model of the dynamics of the pricing kernel $M_{t,t+k}$ and agents' expectations of the dynamics of the future share of factorless income κ_t that we use to account for our measurement of the value of factorless income V_t^{Π}/GVA_t shown in the right panel of Figure 10.

6.1 A General Pricing Formula

We value a claim to factorless income as follows. The price at t relative to output at t for a claim to factorless income at $t + k$ is given as

$$\frac{V_t^{\Pi(k)}}{GVA_t} = \mathbb{E}_t \left[M_{t,t+k} \frac{GVA_{t+k}}{GVA_t} \kappa_{t+k} \right]$$

where $M_{t,t+k}$ is the pricing kernel between periods t and $t+k$ and the value of a claim to all factorless income from $t+1$ on is given by

$$\frac{V_t^{\Pi}}{GVA_t} = \sum_{k=1}^{\infty} \frac{V_t^{\Pi(k)}}{GVA_t}$$

Note that we can write the prices of claims to factorless income at different horizons as

$$\frac{V_t^{\Pi(k)}}{GVA_t} = \frac{P_t^{Y(k)}}{GVA_t} \mathbb{E}_t \left[\kappa_{t+k}\right] + \mathbb{C}ov_t \left(M_{t,t+k} \frac{GVA_{t+k}}{GVA_t}, \kappa_{t+k} - \mathbb{E}_t \left[\kappa_{t+k}\right] \right) \tag{15}$$

where we define the price at t of a claim to output at $t+k$ relative to output at t by

$$\frac{P_t^{Y(k)}}{GVA_t} = \mathbb{E}_t \left[M_{t,t+k} \frac{GVA_{t+k}}{GVA_t} \right]$$

Thus, from equation (15) we have that the price of a claim to factorless income at horizon k relative to output can move for three reasons. First, the price of a claim to output at horizon k relative to output at t given by $\frac{P_t^{Y(k+1)}}{GVA_t}$ might move. Second, the expected factorless income share might move. And third, the risk premium in the covariance term might move.

Note that the pricing equation (15) differs from the standard pricing equation for an asset whose cash flows are always positive in that the covariance term representing risk impacts the *level* of the price of a claim to factorless income. Thus, even if expected factorless income $\mathbb{E}_t[\kappa_{t+k}]$ is positive, the price of a claim to that income can be negative. This property of our pricing model helps us account for observations of the value of factorless income that are below zero corresponding to measurement of values of Tobin's Q that are below one.

Note as well that the covariance term representing risk in equation (15) represents the risk attached to innovations to the share of factorless income. Since, to a first order, changes in the share of factorless income do not impact aggregate output, this risk is not the standard risk to aggregate output or consumption considered in many asset pricing models. Instead, it requires that innovations to the marginal utility of the marginal investor be correlated with innovations to the share of factorless income. Greenwald, Lettau, and Ludvigson (2023) present a model with such a risk premium due to the assumption that the marginal investor derives all of his or her wealth from a claim to cash flows to owners of firms.

6.2 Specification of Process for Factorless Income

We assume that the factorless income share follows an AR(1) with a shifting endpoint

$$\kappa_{t+1} - x_{t+1} = \rho(\kappa_t - x_t) + \epsilon_{\kappa,t+1} \tag{16}$$

$$x_{t+1} = x_t + \epsilon_{x,t+1} \tag{17}$$

These equations imply that

$$\mathbb{E}_t[\kappa_{t+k}] = \rho^k(\kappa_t - x_t) + x_t \tag{18}$$

and

$$\mathbb{E}_t[\kappa_{t+\infty}] = x_t.$$

Thus, x_t is a latent state capturing the long run expected value for the share of factorless income. The gap between the current share κ_t and the long run expected share x_t is expected to close a rate determined by the persistence parameter ρ. Shocks to $\epsilon_{\kappa,t+1}$ have a larger effect on near-term expected factorless income, while shocks to $\epsilon_{x,t+1}$ move expected factorless income permanently:

$$\mathbb{E}_{t+1}[\kappa_{t+k+1}] - \mathbb{E}_t[\kappa_{t+k+1}] = \rho^k(\kappa_{t+1} - x_{t+1}) - \rho^{k+1}(\kappa_t - x_t) + x_{t+1} - x_t = \rho^k\epsilon_{\kappa,t+1} + \epsilon_{x,t+1}$$

This model of the dynamics of the share of factorless income is similar to that used in Kozicki and Tinsley (2001) to model the dynamics of the short-term interest rate. The motivation for this assumed structure is the same in the two applications. Theory suggests that both the short rate and the share of factorless income should be stationary time series. Yet, in the data, both series look like they might be non-stationary and simple valuation models based on either the expectations hypothesis of the term structure or the valuation of factorless income with constant risk adjustments call for very high persistence in these series to account for either the volatility of long-term interest rates or observed volatility of valuations of factorless income. The use of a shifting endpoint delivers this extra persistence.

6.3 No Time Variation in Discount Rates

We will calculate what fluctuations in x_t are needed to account for observed fluctuations in V_t^{Π}/GVA_t under the assumption that the discount rate for claims to factorless income are constant over time. We do not intend to make the claim here that the discount rate for claims to factorless income are constant over time in the data. Instead, we intend this calculation in the spirit of the calculation in Shiller (1981). We expand on the connection between our calculation here and that in Shiller (1981) in greater detail in Appendix section F. We find in this calculation that relatively small fluctuations in the expected factorless income share in the long run are sufficient to account for observed volatility in the value of factorless income.

Given equation (15), pricing factorless income with no time variation in discount rates or risk premia amounts to assuming that the terms $P_t^{Y(k)}/GVA_t$ are constant over time and that the conditional covariance term in that equation at each horizon k is also constant over time. We now discuss conditions under which these terms are indeed constant over time.

Note that the prices $P_t^{Y(k)}$ satisfy the recursion

$$\frac{P_t^{Y(k+1)}}{GVA_t} = \mathbb{E}_t \left[M_{t+1} \frac{GVA_{t+1}}{GVA_t} \frac{P_{t+1}^{Y(k)}}{GVA_{t+1}} \right]$$

with

$$\frac{P_t^{Y(0)}}{GVA_t} = 1$$

The value of a claim to aggregate output from $t+1$ on is given by

$$\frac{P_t^Y}{GVA_t} = \sum_{k=1}^{\infty} \frac{P_{Yt}^{(k)}}{GVA_t}$$

Lemma 1: If the ratio of the price to a claim to output one period ahead to the current value of output, $P_t^{Y(1)}/GVA_t$, is constant over time at a value β, then the price to a claim to output at all future dates relative to current output is also constant over time and is given by

$$\frac{P^Y}{Y} = \frac{\beta}{1-\beta} \tag{19}$$

For the proof of this lemma, see Appendix Section E.

Remark on Lemma 1: By definition

$$\frac{P_t^{Y(1)}}{GVA_t} = \mathbb{E}_t \left[M_{t+1} \frac{GVA_{t+1}}{GVA_t} \right] = \mathbb{E}_t \left[M_{t+1} \right] \mathbb{E}_t \left[\frac{GVA_{t+1}}{GVA_t} \right] + \mathbb{C}ov_t \left(M_{t+1}, \frac{GVA_{t+1}}{GVA_t} \right) \tag{20}$$

Note that

$$\mathbb{E}_t \left[M_{t+1} \right] \mathbb{E}_t \left[\frac{GVA_{t+1}}{GVA_t} \right] = \frac{1}{1+r_t^f} \mathbb{E}_t \left[\frac{GVA_{t+1}}{GVA_t} \right]$$

is a comparison of the expected growth rate of output one period ahead and the one period risk free interest rate. The conditional covariance term in the equation above is a risk premium on a claim to aggregate output one period ahead.

If this conditional covariance term is constant over time, then to have a constant price dividend ratio for a claim to aggregate output, we need to have movements in real interest rates offset by countervailing movements in expected growth rates of aggregate output of the corporate sector so that the term $\frac{1}{1+r_t^f}\mathbb{E}_t\left[\frac{GVA_{t+1}}{GVA_t}\right]$ is constant over time. To the extent that this conditional covariance term moves over time, the assumption that $P_t^{Y(1)}/GVA_t$ is constant over time is equivalent to assuming that movements in the gap between the risk free interest rate and the expected growth rate of output correspond to offsetting movements in the risk premium on a claim to output one period ahead.

Now we turn to developing conditions under which the conditional covariance term in equation (15) at each horizon k is also constant over time.

Lemma 2: Assume that $P_t^{Y(1)}/GVA_t$ is constant over time and equal to β, and that for all k and all t

$$\mathbb{C}ov_t\left(M_{t+1}\frac{GVA_{t+1}}{GVA_t}, \epsilon_{\kappa,t+1}\right) = C$$

and

$$\mathbb{C}ov_t\left(M_{t+1}\frac{GVA_{t+1}}{GVA_t}, \epsilon_{x,t+1}\right) = D$$

That is, assume that the covariance of innovations to the expected factorless income share and the product of the pricing kernel and output growth are constant over time. Then, for $k \geq 1$,

$$\frac{V_t^{\Pi(k)}}{GVA_t} = \beta^k\mathbb{E}_t[\kappa_{t+k}] + \beta^{k-1}\left(\sum_{s=0}^{k-1}\rho^s C + kD\right)$$

For the proof of this lemma, see Appendix E. We provide an example of a full pricing kernel for which the assumption is satisfied in that appendix.

When the conditions of Lemma 2 are satisfied, we can write the value of a claim to factorless income as

$$\frac{V_t^{\Pi}}{GVA_t} = \left[\frac{\beta\rho}{1-\beta\rho}\right](\kappa_t - x_t) + \frac{\beta}{1-\beta}x_t + \phi \tag{21}$$

where

$$\phi = \sum_{k=1}^{\infty}\beta^{k-1}\left(\left(\frac{1-\rho^k}{1-\rho}\right)C + kD\right).$$

We explore this quantitative implication of our equation (21) for valuing factorless income in the next subsection.[20]

[20]Note that the assumptions that we need for our two lemmas do not require that all expected excess returns be constant. In Appendix section E, we present a flexible specification of an essentially affine pricing kernel with time-varying volatility of the stochastic discount factor as in Campbell (2018) Chapter 8.3.3 that would allow for a rich model of the term structure of interest rates with time-varying expected excess returns to holding long term bonds and a full set of time-varying risk premia on other claims as described in Lustig, Van Nieuwerburgh, and Verdelhan (2013) and Jiang et al. (2022). We show how to derive the implications of this model for the term structure of interest rates and for the prices for claims to output at different

6.4 Our Valuation Exercise

We use equation (21) together with our measurement of κ_t from Section 5.4 to back out estimates of the time series for x_t needed to reconcile equation (21) with the data on the value of factorless income relative to output shown in the right panel of Figure 10. We will evaluate the outcome of this measurement in three ways.

First, we will explore the implied volatility of the time series for x_t. If our model's measurement of x_t has no variability, then the data on the valuation of factorless income can be reconciled simply as a result of observed fluctuations in the current share of factorless income projected to decay to a constant unconditional mean at rate ρ. If the estimated series for x_t is highly variable, we regard it as improbable that expected fluctuations in the share of factorless income alone can reasonably account for observed valuations of factorless income.

Second, we evaluate the extent to which the innovations $\epsilon_{x,t+1}$ and $\epsilon_{\kappa,t+1}$ are themselves not forecastable. Our valuation model takes these to be expectation errors in the dynamics of free cash flow. But, since we back out values of x_t period-by-period from equation (21), there is nothing in our measurement procedure to guarantee that these innovations are, in fact, expectation errors. Hence, a finding that these innovations are predictable would indicate misspecification of our valuation model.

The third way in which we will evaluate the exercise is by considering whether the fluctuations that we uncover in x_t do in fact predict subsequent growth in factorless income's share of value added. In particular, our process for κ_t implies a relationship between x_t and expected future factorless income as given by equation (18). Once we have a series for x_t we can measure whether subsequent factorless income evolves, on average, consistently with this expression. If it does, that would be evidence in favor of our model specification, and evidence that observed fluctuations in valuations reflect rational expectations of future factorless income growth. On the other hand, if the x_t series that we uncover does not forecast future factorless income growth, then that would cast doubt on our process for κ_t. Such a finding would favor alternative theories in which fluctuations in valuations are driven either by irrational waves of optimism or pessimism about future income growth, or alternatively by time variation in discount rates or risk premia.

horizons using standard calculations. We also develop a formula for pricing claims to factorless income when the pricing kernel and output growth are conditionally lognormal but innovations to the factorless income share are normal. The assumptions that we have made in our previous two lemmas amount to parameter restrictions in this model that could be evaluated in an estimation exercise. We do not intend to claim that these conditions hold in the data. We leave evaluation of that question to future research.

6.5 Calibrating Valuation Parameters

Equation (21) has three parameters. The first is the value for β or, equivalently, the price dividend ratio for a claim to aggregate output P^Y/Y in equation (19). The second is the persistence of the factorless income share ρ. The third is the constant risk premium ϕ on a claim to factorless income due to the covariances of innovations to κ_t and x_t with the product of the pricing kernel and aggregate growth.

We are not aware of any direct measurement of the price-dividend ratio P^Y/Y for a claim to the future aggregate output of the U.S. corporate sector. Lustig, Van Nieuwerburgh, and Verdelhan (2013) argue that the price dividend ratio for a claim to aggregate consumption is quite high. Based on that analysis, we consider a baseline value of $P^Y/Y = 50$ corresponding to $\beta = \frac{50}{51}$. We will also consider alternative values of $P^Y/Y = 25$ and $P^Y/Y = 100$.[21]

We estimate ρ and ϕ as follows. Equation (21) can be rearranged to give

$$\kappa_t - x_t = \frac{1}{\frac{\beta}{1-\beta} - \frac{\beta\rho}{1-\beta\rho}} \left[\frac{\beta}{1-\beta}\kappa_t - \frac{V_t^{\Pi}}{GVA_t} + \phi \right]. \tag{22}$$

We substitute this expression into equation (16) and estimate ρ and ϕ by least squares by regressing $\frac{\beta}{1-\beta}\kappa_{t+1} - \frac{V_{t+1}^{\Pi}}{GVA_{t+1}}$ on a constant and the same variable at date t. The slope coefficient provides a direct estimate of ρ, while the constant corresponds to $\phi(\rho - 1)$. This procedure gives $\rho = 0.8745$ and $\phi = -0.943$. One way to interpret this value for ϕ is that when current factorless income and long run expected factorless income are both zero ($\kappa_t = x_t = 0$) investors need to be paid 94 percent of gross value added to be willing to hold factorless income firms.

Recall that given our choice for α and tax parameters, we can construct a series for factorless income's share κ_t from equation (10) and for the associated valuation $\frac{V_t^{\Pi}}{GVA_t}$ from equation (14). Then, given values for β, ρ and ϕ we can construct a series for the latent expectation variable x_t that supports these valuations from our valuation equation (21 or 22).

[21] We do note that in our data, the difference between the realized growth of nominal corporate after-tax gross value added and the one-year risk free nominal rate at the end of the prior year has a mean of 2.9% and a standard deviation of 8.5% in our data from 1929-2022. Thus, if the price of a claim to output one period ahead relative to current output is one, corresponding to a value of $P^Y/Y = \infty$, then the Sharpe Ratio on that claim would be 0.34, since realized growth in output in excess of the short rate would correspond to the realized excess returns on that claim to output one period ahead. In contrast, if the price of a claim to output one period ahead relative to current output were 0.98, corresponding to a value of $P^Y/Y = 50$ as in our baseline case, then the Shape Ratio on that claim to output one period ahead would be 0.58, which seems quite high relative to standard estimates of the Sharpe Ratio for the stock market. Regardless, we take this as a baseline value. In sum, output growth in our data is not nearly as volatile as returns on enterprise value or equity and thus a claim to output one period ahead should not have such a high risk premium, leading to a high value of P^Y/Y.

With $\alpha = 0.29$, the unconditional sample mean value of κ_t from equation (10) is 0.0298. The average value for x_t that emerges given our other parameter choices is very similar, at 0.0303.

6.6 Baseline Results

Consider first the results of our baseline measurement exercise with $\alpha = 0.29$, $P^Y/Y = 50$, $\rho = 0.8745$, and $\phi = -0.943$. We plot the implied dynamics of the long-run factorless income share x_t in Figure 11. We see in this figure that the implied time series for x_t is relatively smooth over time. That result obtains because, with a high value for β, the coefficient on x_t in equation (21) is large. Thus, small movements in the expected share of factorless income in the long run have a powerful impact on the implied present value of factorless income. In particular, the coefficient on κ_t in valuation equation (21) is equal to 6.0 while the combined coefficient on x_t in this equation is 50-6=44. Thus, a movement in the the expected share of factorless income of one percentage point in the long-run accounts for a movement in the value of factorless income relative to output of 44 percentage points.

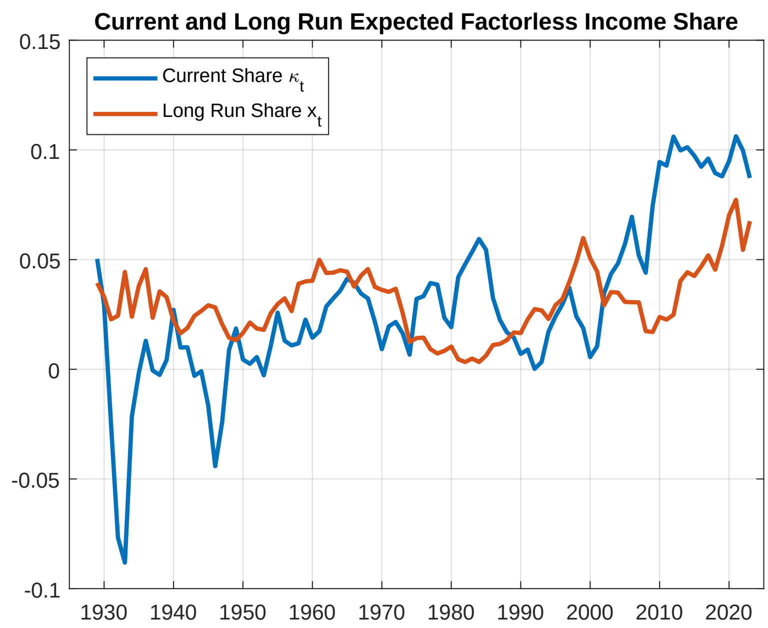

Figure 11: Share of Factorless Income κ_t in U.S. Corporate Gross Value Added (blue) and Long-Run Expected Factorless Income Share x_t (red).

6.7 Is Cash Flow Really Predictable?

Figure 11 suggests that long run expected factor x_t does in fact predict subsequent factorless income κ_t. In the first decades of our sample, x_t exceeds κ_t, indicating expected future growth in the factorless income share. And the factorless income share does indeed gradually drift up. The 1990s is another period when x_t exceeds κ_t, and factorless income subsequently grew strongly in the 2000s. Note that at the end of our sample, long run expected factorless income is below the current level. Thus, interpreted through the lens of the model, current cash flow is more than sufficient to justify observed corporate valuations in 2023.

To investigate free cash flow growth predictability more systematically, consider again equation (22). That equation shows a linear model relationship between a price minus cash flow statistic, $\frac{V_t^{\Pi}}{GVA_t} - \frac{\beta}{1-\beta}\kappa_t$ and the latent unobserved gap between long-run expected dividends relative to consumption and the current value of that ratio, $x_t - \kappa_t$. If the observed price per share at some date t is high relative to current factorless income, the model interpretation is that expected long run factorless income must be high relative to current factorless income. Equation (16) indicates that investors expect this gap to narrow over time, as long as $\rho < 1$. More specifically, model investors expect κ_t to gradually rise and catch up to x_t, and they expect no change in x_t given the unit root process in equation (17). Thus, a high current valuation minus cash flow differential signals rapid expected future cash flow growth. The model-predicted dividend growth between t and $t + s$ is given by particular, given equations (16) and (17),

$$\mathbb{E}_t\left[\kappa_{t+s}\right] - \kappa_t = \left(1 - \rho^s\right)\left(x_t - \kappa_t\right). \tag{23}$$

To test whether these cash flow growth predictability properties are consistent with the data we run some simple forecasting regressions. In particular, for different forecasting horizons s, we regress growth between t and $t + s$ in the share of factorless income on $x_t - \kappa_t$. Thus, we estimate coefficients β_s for the model

$$\kappa_{t+s} - \kappa_t = \alpha_s + \beta_s\left(x_t - \kappa_t\right) + \epsilon_{t+s}. \tag{24}$$

We also run a similar regression but with $x_{t+s} - x_t$ on the left-hand side.[22]

Figure 12 plots the results. The blue and red lines shows that our latent gap variable $x_t - \kappa_t$ is strongly predictive of future growth in free cash flow to factorless income. Average realized growth in free cash flow exceeds the model-implied rational expectations value $(1-\rho^s)(x_t-\kappa_t)$ (the purple line) at horizons up to around 6 years, but falls short of that value at longer horizons.

[22]There is no constant in the theoretical relationship (23). We have run the forecasting regressions with and without the constant term α_s.

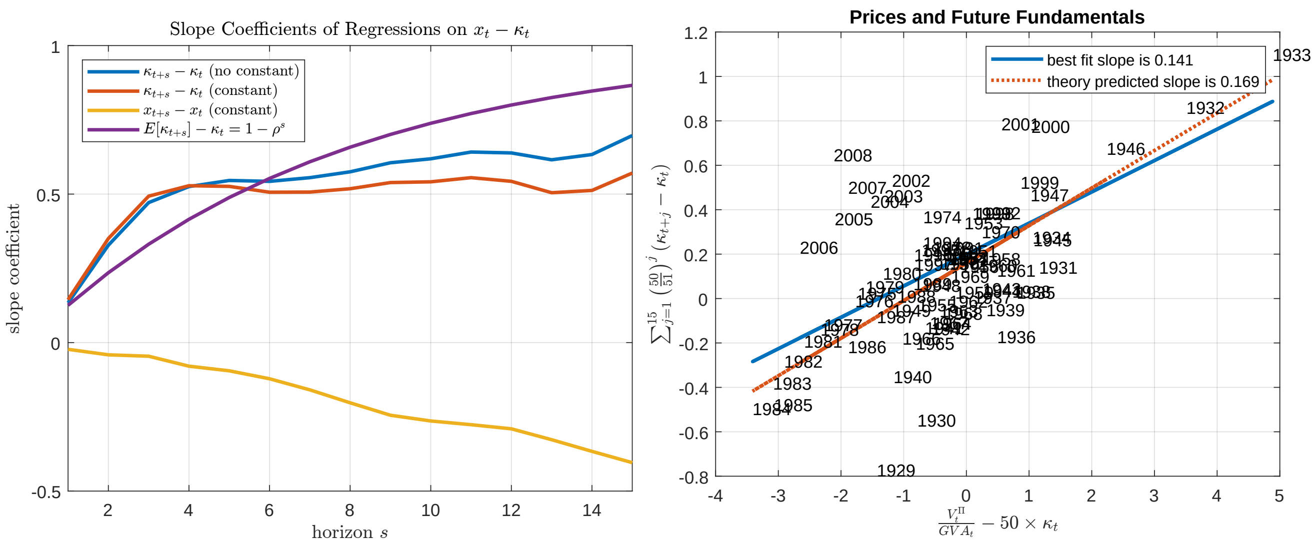

Figure 12: Left Panel: Forecasting Regressions at Different Horizons s. The right-hand side variable is $x_t - \kappa_t$. The left-hand side variable for the blue and red lines is $\kappa_{t+s} - \kappa_t$. The plots shows the slope coefficients β_s when equation (24) is estimated without a constant (blue line) and with a constant (red line). The purple line shows the model-predicted growth, $E[\kappa_{t+s}] - \kappa_t = 1 - \rho^s$. The red line shows the slope coefficient when the left-hand slide variable is $x_{t+s} - x_t$. For every horizon s, the right-hand side regressor runs from 1929 to 2008. Right Panel: Factorless Income Predictability. The y-axis variable is cumulative discounted growth in factorless income over a 15 year horizon starting from the year shown. The x-axis variable is factorless income valuation minus 50 times the current factorless income share. The starting year runs from 1929 to 2008.

The yellow line in the plot indicates that $x_t - \kappa_t$ does not predict growth in x_t at short horizons, but at longer horizons high values for $x_t - \kappa_t$ appear to predict declines in x_t, which ought to be unpredictable given our unit root assumption. We will explore further whether this forecast is significant in future revisions of this paper.

We next illustrate cash flow predictability in a more direct way. For each year in our sample, we record the valuation of factorless income minus 50 times the current factorless income share: $\frac{V_t^{\Pi}}{GVA_t} - 50\kappa_t$. We also record the discounted sum of factorless income growth over the subsequent 15 years, $\sum_{j=1}^{15} \left(\frac{50}{51}\right)^j (\kappa_{t+j} - \kappa_t)$. The right panel of Figure 12 plots the corresponding scatter plot, with each year t labelled. The blue line shows the least squares best fit. The red line instead plots $\sum_{j=1}^{15} \left(\frac{50}{51}\right)^j (\mathbb{E}_t [\kappa_{t+j}] - \kappa_t)$ on the y axis, where $\mathbb{E}_t [\kappa_{t+j}]$ is given by equation (23). It is clear from this plot that this value minus cash flow statistic does predict subsequent growth in the share of factorless income. And overall, the extent of predictability in the data is very similar to the extent of theoretical predictability embedded in our calibrated value for the convergence parameter ρ. The biggest outliers are

1929, when valuations are high relative to subsequent factorless income growth, and 2008, when valuations are low relative to subsequent factorless income growth.

We conclude that growth in free cash flow from factorless income has a large predictable component, and that fluctuations in observed valuations of claims to factorless income can reasonably be interpreted as reflecting rational changes in expectations about that growth.

6.8 Sensitivity with Respect to Discount Rate

The ratio of the price of a claim to aggregate output relative to current output plays an important role in our analysis as it impacts the size of the coefficients on κ_t and $x_{\kappa t}$ in equation (21). We now conduct a sensitivity analysis of our measurement with respect to this key parameter β. We present the results of this exercise in Figure 13. When we conduct our sensitivity analysis to alternative choices of P^Y/Y below, we do not change the parameter α, but we re-estimate ρ and ϕ.

In the left panel of this figure, we show in red the estimate of x_t when $\frac{P^Y}{Y} = 25$, rather than our baseline value of $\frac{P^Y}{Y} = 50$. In this case, we estimate $\rho = 0.840$ and $\phi = -0.146$. Intuitively, the model infers a less negative risk premium term to rationalize the same observed sequence for κ_t with a lower value for β. We see in this figure that the implied fluctuations in the expected long-run share of factorless income needed to account for the data on the valuation of factorless income are roughly twice as large as in our baseline case. This implication of our model arises because, with these parameters, the coefficient on $x_{\kappa t}$ in the valuation equation is less than half its value with our baseline parameters.

In the right panel of this figure, we show in red the estimate of x_t when $\frac{P^Y}{Y} = 100$. In this case, we estimate $\rho = 0.899$ and $\phi = -2.603$. Intuitively, the model infers a more negative risk premium term to rationalize the same observed sequence for κ_t with a higher value for β. We see in this figure that the fluctuations in the long run expected value of factorless income needed to account for the data on the fluctuations in the valuation of that income are quite small. This is because, with these parameters, the coefficient on x_t in equation (21) is now 91.9. As a result, in this case, we find substantially less variability in our estimate of x_t than we did in Figure 11.

The main implication of these results is that if the price-dividend ratio for a claim to aggregate output is large, then relatively small shifts in agents' expectations for the share of factorless income in the long run are sufficient to account for the observed variability of the value of a claim to factorless income relative to output. Thus, the answer to the question of whether there is "excess volatility" of the market valuation of U.S. corporations relative to the volatility of their cash flows to owners of these firms is contingent on what assumptions one makes about this price-dividend ratio of a claim to aggregate output.

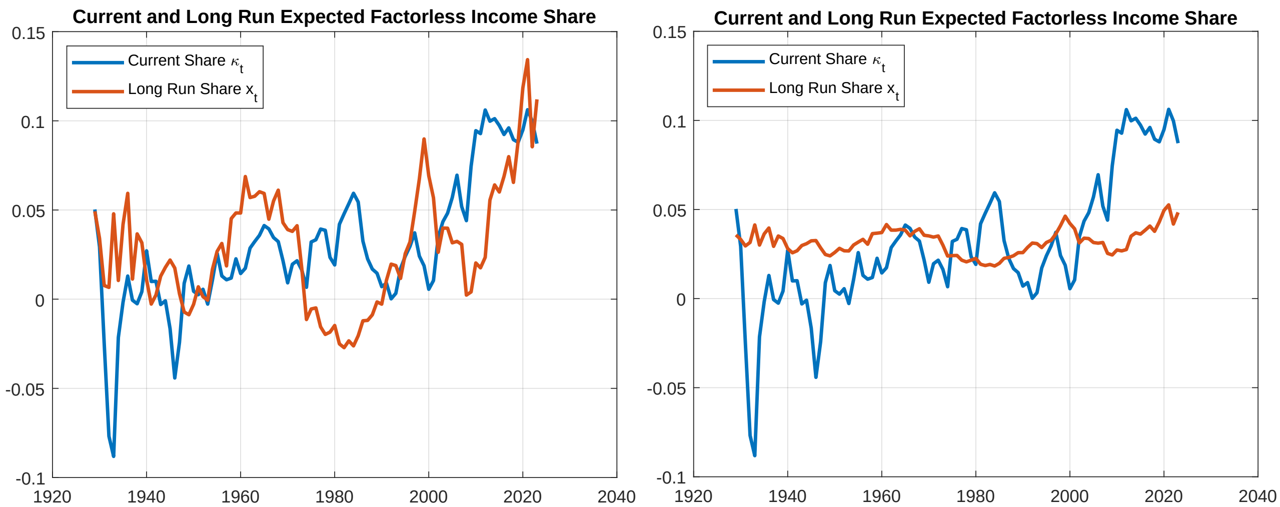

Figure 13: Left Panel: Share of Factorless Income κ_t in U.S. Corporate Gross Value Added (blue) and Expected Long-Run Factorless Income Share x_t (red). Expected Long Run Share Calculated at $\frac{P^Y}{Y} = 25$. Right Panel: Share of Factorless Income κ_t in U.S. Corporate Gross Value Added (blue) and Expected Long-Run Factorless Income Share x_t (red). Expected Long Run Share Calculated at $\frac{P^Y}{Y} = 100$.

7 Returns

We now turn to an exploration of returns. Expected returns to a real risk free bond, to a real GDP bond, to capital, and to factorless income firms satisfy, respectively:

$$
\begin{aligned}
1 &= (1+r_t^f)\mathbb{E}_t\left[M_{t,t+1}\right], \\
1 &= \mathbb{E}_t\left[M_{t,t+1}(1+r_{t+1}^Y)\right], \\
1 &= \mathbb{E}_t\left[M_{t,t+1}(1+r_{t+1}^K)\right], \\
1 &= \mathbb{E}_t\left[M_{t,t+1}(1+r_{t+1}^\Pi)\right].
\end{aligned}
$$

In constructing our valuation model for factorless income, we imposed assumptions on the pricing kernel $M_{t,t+1}$ such that the equilibrium price of a real GDP bond is $P_t^Y = \frac{\beta}{1-\beta}GVA_t$ and thus the realized return is

$$1 + r_{t+1}^Y = \frac{P_{t+1}^Y + GVA_{t+1}}{P_t^Y} = \frac{1}{\beta}\frac{GVA_{t+1}}{GVA_t}.$$

The realized return to factorless income is

$$1 + r_{t+1}^{\Pi} = \frac{GVA_{t+1}}{GVA_t} \frac{\frac{V_{t+1}^{\Pi}}{GVA_{t+1}} + \kappa_{t+1}}{\frac{V_t^{\Pi}}{GVA_t}}.$$

Substituting in valuations V_t^{Π}/GVA_t from equation (21) gives

$$1 + r_{t+1}^{\Pi} = \frac{GVA_{t+1}}{GVA_t} \left[\frac{\frac{\beta\rho}{1-\beta\rho}(\kappa_{t+1} - x_{t+1}) + \frac{\beta}{1-\beta} x_{t+1} + \phi + \kappa_{t+1}}{\frac{\beta\rho}{1-\beta\rho}(\kappa_t - x_t) + \frac{\beta}{1-\beta} x_t + \phi} \right].$$

From the laws of motion for κ_t and x_t, the expected return is

$$\mathbb{E}_t \left[1 + r_{t+1}^{\Pi}\right] = E_t \left[\frac{GVA_{t+1}}{GVA_t} \right] \frac{\Phi_t}{\beta},$$

where

$$\Phi_t = \frac{\rho(\kappa_t - x_t) + \frac{(1-\beta\rho)}{1-\beta} x_t + (1 - \beta\rho)\,\phi}{\rho(\kappa_t - x_t) + \frac{(1-\beta\rho)}{1-\beta} x_t + \frac{1-\beta\rho}{\beta}\phi}.$$

Note that if the risk premium term $\phi = 0$, then $\Phi_t = 1$ and the expected return to factorless income firms is identical to the expected return to a real GDP bond. If $\phi < 0$ and $\beta < 1$ (as in our baseline calibration) then Φ_t constitutes a wedge between the expected return to factorless income firms and the expected return to GDP bonds. If the price V_t^{Π} is positive, then both the numerator and the denominator in the expression for Φ_t are positive, in which case $\phi < 0$ implies $\Phi_t > 1$. Thus, the expected return to factorless income firms exceeds the return to a GDP bond. Intuitively, if κ_t share risk is priced, then investors demand higher expected returns to hold claims to it.

The gross realized return to capital is given by

$$\begin{aligned} 1 + r_{t+1}^{K} &= \frac{V_{t+1}^{K} + FCF_{t+1}^{K}}{V_t^{K}} \\ &= \frac{1}{(1 - \lambda\tau_t^c)} \left[(1 - \tau_{t+1}^c) \left(\frac{\frac{R_{t+1}^K K_{t+1}}{GVA_{t+1}} \frac{GVA_{t+1}}{GVA_t}}{\frac{Q_t K_{t+1}}{GVA_t}} - \frac{Q_{t+1}}{Q_t} \delta_{t+1} \right) + (1 - \lambda\tau_{t+1}^c) \frac{Q_{t+1}}{Q_t} \right]. \end{aligned}$$

where

$$\frac{R_{t+1}^K K_{t+1}}{GVA_{t+1}} = \alpha \left[(1 - \tau_{t+1}^s) - \frac{\left(\kappa_{t+1} - \tau_{t+1}^L\right)}{(1 - \tau_{t+1}^c)} \right]$$

Computing the expected value of this expression, $1 + \mathbb{E}_t\left[r_{t+1}^K\right]$, requires specifying a joint stochastic process for $\left(\tau_t^c, \tau_t^s, \tau_t^L, \delta_t, \frac{Q_{t+1}}{Q_t}, \frac{GVA_{t+1}}{GVA_t}\right)$. We assume all these variables are independent, and that for every tax parameter $\mathbb{E}_t\left[\tau_{t+1}\right] = \tau_t$. We assume $\mathbb{E}_t\left[\delta_{t+1}\right] = \delta_t$ and $\mathbb{E}_t\left[\frac{Q_{t+1}}{Q_t}\right] = \bar{g}_Q$, where $\bar{g}_Q$ is the sample average gross growth rate for the relative price of investment goods. The expected value for κ_{t+1} is equal to $\rho\kappa_t + (1-\rho)x_t$. We allow expected real growth in value added to vary over time, and we set the conditional expectation $\mathbb{E}_t\left[\frac{GVA_{t+1}}{GVA_t}\right]$ at each date t such that the expected return to capital is equal to the expected return to a GDP bond.[23]

$$1 + \mathbb{E}_t\left[r_{t+1}^K\right] = 1 + \mathbb{E}_t\left[r_{t+1}^Y\right] = \frac{1}{\beta}\mathbb{E}_t\left[\frac{GVA_{t+1}}{GVA_t}\right]. \tag{25}$$

The left panel of Figure 14 plots realized returns to capital alongside expected returns, computed as just described. As one might expected, expected returns track realized returns closely. The main source of volatility in realized returns to capital is valuation effects associated with changes in the relative price Q_t. Expected returns were very high around the end of World War II, and subsequently declined fairly steadily. The right panel of the figure plots expected real growth in gross value added against realized growth. Absent an expected return differential between capital and GDP bonds, expected growth in real value added is simply $\beta = 0.98$ times the expected return to capital. Thus, we infer expected growth in value-added declined from around 5 percent per year in the 1960s to around 2.5 percent per year at the end of the sample period. The series for expected growth inferred from returns tracks actual growth fairly closely, although the average return is slightly higher.

[23] The logic underlying this choice is that neither GDP bonds nor physical capital are especially risky assets – one could imagine a small relative risk premium in either direction. We have also constructed a series for the expected return to capital assumed a constant expected growth rate for gross value added. The resulting series looks near identical, which reflects the fact that the return to capital is not very sensitive to output growth.

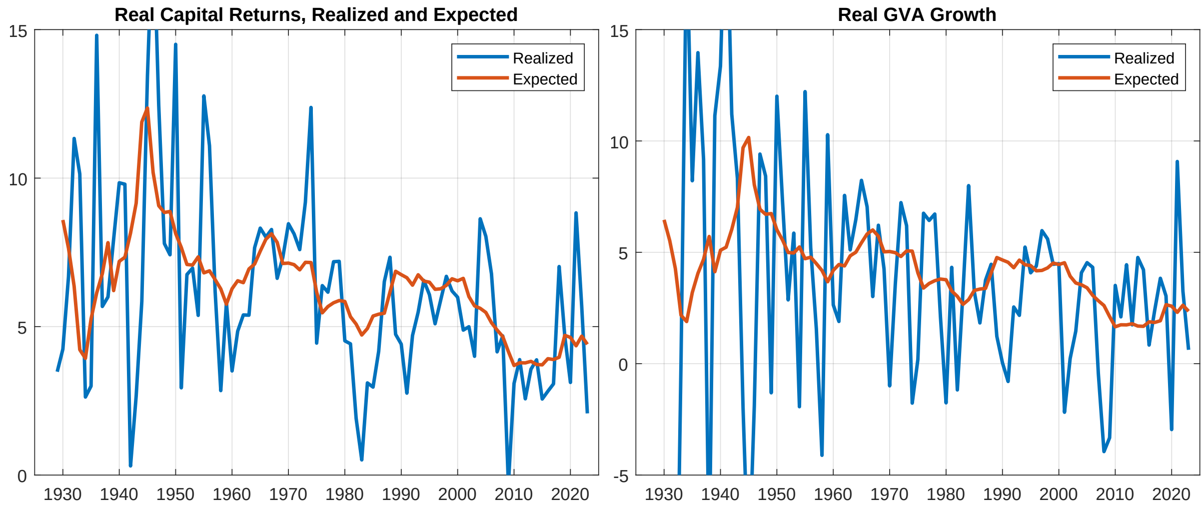

Figure 14: Left Panel: Realized and Expected Real Returns to Capital. Right Panel: Realized and Expected Growth in Real Gross Value Added.

Figure 15 plots expected returns to capital and expected returns to enterprises, where the latter are computed as

$$\mathbb{E}_t\left[r_{t+1}^V\right] = \frac{V_t^K}{V_t}\mathbb{E}_t\left[r_{t+1}^K\right] + \frac{V_t^\Pi}{V_t}\mathbb{E}_t\left[r_{t+1}^\Pi\right]$$

The expected return to enterprises tracks the expected return to capital closely. The fact that the expected return to enterprises exceeds the expected return to capital reflects our choice for the capital share parameter α. Recall that a higher value for α implies that a larger share of income accrues to capital, reducing returns to capital. At the same time, the choice for α does not affect the total return to enterprise value, which is measured straight from the data. Our baseline choice of $\alpha = 0.29$ implies that the average log return to enterprises minus the average log return to capital is 0.0102.

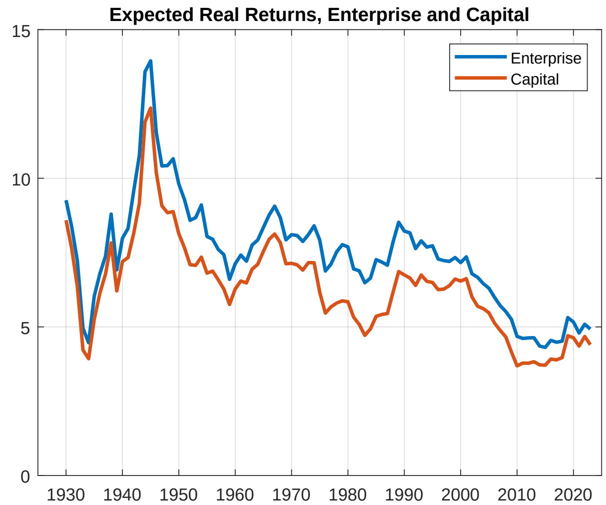

Figure 15: Expected Real Returns to Enterprises and to Capital.

Given a common pricing kernel, the expected excess return to physical capital relative to a risk free bond is given by

$$\mathbb{E}_t\left[r^K_{t+1}\right] - r^f_t = -(1 + r^f_t)\mathbb{C}ov_t\left(M_{t,t+1}, (1 + r^K_{t+1})\right) \tag{26}$$

where the term on the right side of this equation is the risk premium on physical capital. If this risk premium has been relatively stable over time, one would expect a stable differential between expected returns to capital and the risk free rate.

Figure 16 plots the difference between the expected return to capital and the rate on 3 month T-Bills. Returns to capital greatly exceeded safe returns during the 1940s, but the return differential appears fairly stable at around 5 percentage points during the 1960s and 70s. In the early 1980s the differential fell to around zero, as the Federal Reserve pushed up short term rates to combat inflation. But from the 1990s onward, the differential again appears fairly stable at around 5 percent.

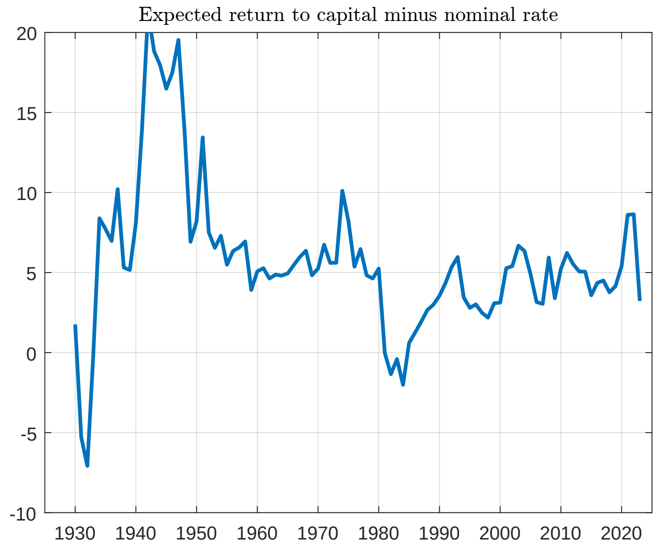

Figure 16: Expected Return to Capital Minus Short Run Treasury Rate. 1929-2023.

We conclude that our decomposition of cash flow between income to capital and factorless income yields a plausible time path for the after-tax return to capital, with a downward trend in expected returns over time that broadly tracks measured declines in safe rates. The return to capital declines because free cash flow to capital has been declining (Figure 9) while at the same time the price of capital has been gradually rising from around 1970 onward (Figure 10). At the same time, the calibrated model implies a path for expected returns to factorless income firms such that the total expected return to corporate enterprises closely tracks the expected return to capital, with a positive and relatively stable risk premium.

Thus, the observed series for corporate free cash flow, for enterprise value and for capital investment are all mutually consistent with a model in which all expected returns have been relatively stable over time, and have all trended downward over time.

8 Conclusion

In this paper we have explored the hypothesis that the data in the Integrated Macroeconomic Accounts form a useful unified data set for work in macroeconomics and finance. To illustrate the potential utility of this data set, we first explored the correspondence between measures of returns and valuation in these IMA data with corresponding measured obtained from public firm data. We then used these data to revisit some important questions in macrofinance regarding the drivers of the volatility of the market valuation of U.S. corporation using both Campbell-Shiller regression analysis and a simple valuation model comparable to that in Shiller (1981). Finally, we developed a simple variant of a standard stochastic growth model to provide an accounting of the relationship between the realized returns to physical capital and financial claims on firms and used that to raise a new puzzle regarding the observed trends in returns on physical capital in the United States.

This paper is structured as an exploratory tour of the rich information in the Integrated Macroeconomic Accounts. Clearly more research will be needed to resolve any one of the issues we have raised using these data. It is our aim to do that more focused study in subsequent papers. We hope readers of this paper will be motivated to use the Integrated Macroeconomic Accounts to address these and other macro-finance questions in their research so that we might finally have a full reconciliation of macroeconomics and finance based on a common set of data.

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Appendices

A Data Appendix

In this appendix we list the sources for the data used in this paper. We make reference to four main sources of information. There is often considerable overlap between these data sources.

A.1 Aggregate data

- Integrated Macroeconomic Accounts (IMA) Tables S5 and S6 for the Nonfinancial Corporate and Financial Sectors respectively. These tables present flows from 1946-2022 and end of year balance sheets from 1945 to 2022. These tables can be found towards the back of the publication Z1 *Financial Accounts of the United States* (previously known as the Flow of Funds). They can also be found on the website of the Bureau of Economic Analysis.

- National Income and Product Accounts (NIPA) Table 1.14. This table offers annual data on gross value added and the breakdown of income into components for the corporate sector from 1929 through 2022. We also refer to other tables from the NIPA and refer to them with this abbreviation.

- Fixed Assets (FA) Tables 6.1, 6.4, and 6.7 which offer data on investment, consumption of fixed capital (depreciation), and year-end capital stocks for the non-financial corporate and financial sectors from 1929 to the present.

- Various tables from the *Financial Accounts of the United States* which we refer to by the abbreviation FOF and the table number.

We now describe our specific data sources.

The following series for the corporate sector 1929-2023 are taken from NIPA Table 1.14. These data series are also available broken down for the non-financial corporate and financial sectors separately on NIPA Table 1.14 and on IMA Tables S5 and S6. These tables are updated on different schedules, so the source with the most up to date data depends on the time of year. Small differences between these two data sources may exist due to different accounting standards for the NIPA and the IMA. We list the line and table numbers for these series below.

- Gross Value Added. NIPA Table 1.14 Line 1. See also IMA Tables S5 FA106902501.A and S6 FA796902505.A

- Tax Payments are the sum of three lines from NIPA Table 1.14. These are line 7, Taxes on production and imports less subsidies, line 10, Business current transfer payments (net), and line 12, Taxes on corporate income. See also IMA Tables S5 FA106240101.A, FA106403001.A, FA106220001.A and S6 FA796240101.A, FA796403005.A, FA796220001.A

- Compensation of Employees. NIPA Table 1.14 Line 4. See also IMA Tables S5 FA106025005.A, FA796025005.A
- Consumption of Fixed Capital. NIPA Table 1.14 Line 2. See also IMA Tables S5 FA106300003.A, FA796330081.A

We obtain data on investment expenditures (gross fixed capital formation) by the corporate sector from two sources listed below. Small differences between the data on Fixed Assets Table 6.7 and IMA Tables S5 and S6 for the period for which they overlap are due to different accounting standards for the two accounts.

- Investment 1929 - 1945. Fixed Assets Table 6.7 line 2
- Investment 1946 - 2022. The sum of IMA Tables S5 line FA105019085.A and S6 line FA795013005.A

We obtain data on the reproduction value of the capital stock in the corporate sector from two sources listed below. It is important to note that the value of nonfinancial assets listed on the balance sheets of Tables S5 and S6 include measures of the value of land, which we exclude from our model. Thus, we do not use those measures. Instead we use the following sources that are restricted to fixed assets.

- Capital 1929 - 1944. Fixed Assets Table 6.1 line 2
- Capital 1945 - 2023. FOF Table L4. Sum of lines FL105015085.A and FL795013865.A

We obtain data on enterprise value of the corporate sector from two sources.

- For the period 1945 - 2023, we use balance sheet data from IMA Tables S5 and S6. These series are constructed for the Nonfinancial corporate sector as the difference between the line Total Liabilities and Net Worth minus the line titled Financial Assets. The construction is the same for the Financial Sector using Table S6. These series for enterprise value are reported on FOF Table B1 in lines LM102010405.A and LM792010405.A. We use these series from B1.
- For the period 1929 - 1944 we use data from the 1945 Statistics of Income Part 2 available here https://www.irs.gov/pub/irs-soi/45soireppt2ar.pdf. We use data from Table 20 on page 420 of this document (page 425 of the PDF). For financial assets, we use Total Assets on line 9 less Capital Assets on line 7. For liabilities, we use Total Liabilities on line 21 less Capital Stock Common on line 17. For the market value of equity, we use the total market capitalization for the CRSP Value Weighted Index.

We use two sources of data on the market value of corporate equities. These are

- 1929-1944: CRSP Value Weighted Index Total Market Capitalization

- 1929-1944: FOF Table L224 Nonfinancial Corporate Equities LM103164105.A for both public and closely held and LM103164115.A for public alone plus Financial Sector Corporate Equities LM793164105.A less equities issued by Closed End Funds from FOF Table L123 LM554090005.A and by ETFs from FOF Table L124 FA564090005.A.

We use two sources of data on dividend payments. We pay particular attention to distinguishing between dividends as reported in the IMA (and other places) and monetary dividends paid. One of the big distinctions between these two concepts concerns the treatment of dividends on foreign direct investment which are an accounting entry and not a measure of dividends paid. Note, however, that our measure of dividends does include cash dividends paid on foreign direct investment in the U.S. Our sources are

- 1929 - 1945. NIPA Table 7.16 line 31 (Dividends paid in cash or assets, IRS), plus line 32 (Post tabulation amendments and revisions), less line 33 (Dividends paid by Federal Reserve Banks).
- 1946 - 2022. NIPA Table 7.10 line 2 (Monetary Dividends Paid Domestic Corporate Business) less line 4 (Paid by Federal Reserve Banks).

We use two sources of data for returns on corporate equities.

- 1929-1945. Returns without dividends are set equal to returns without dividends on the CRSP Value Weighted Total Market Index. The dividend return is computed as the ratio of dividends paid in year $t+1$ to the value of corporate equities in year t.
- 1946 - 2022. Returns without dividends are set equal to the ratio of the sum from Table S5 of revaluations of nonfinancial corporate equities FR103164105.A and non financial foreign direct investment in the U.S. FR103192105.A and from Table S6 revaluations of financial corporate equities FR793164105.A and financial foreign direct investment in the U.S. FR793192105.A to the sum of the corresponding levels of these variables at the end of the previous year: from S5 LM103164105.A + FL103192105.A and from S6 LM793164105.A + LM793192105.A. Returns with dividends adds to this return the ratio of dividends in year t to the value of corporate equities plus FDI equity in the U.S. in year $t-1$.

A.2 Compustat data

For producing the Compustat lines in figures 5 and 6 we restrict the Compustat sample as follows. First we select only firms incorporated in the United States, we then drop observations for firms which report their 10k in financial services format, observations for which there is no year information or for which a firm year is duplicated. That leaves us with a sample of 316562 firm/year observations over the 1988-2023 period. Free cash flow is computed (following Adame et al. 2023) as Operating Activities–Net Cash Flow (OANCF) minus capital expenditures (CAPX). Enterprise value is computed as Total Market Value (MKVALT) plus Total Liabilities (LT) minus current assets total (ACT), which includes cash and other short term investments, receivables, inventories, and other current assets.

B Data Statistics from 1945-2022

In this appendix, we include additional data plots and focus on data series starting in 1945.

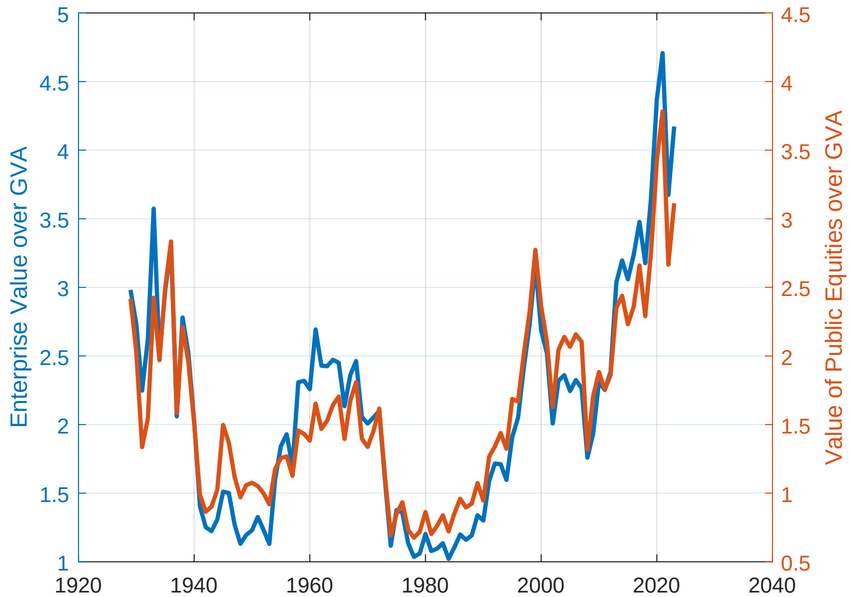

Figure B.1: Enterprise Value (left axis) and the Market Value of Corporate Public Equities (right axis) over Gross Value Added. 1929-2023

The Integrated Macroeconomic Accounts start with measures of end of year balance sheet items in 1945. In this section, we report statistics computed using only the data from these accounts.

Table B.1: Mean and Standard Deviation of Real Log Returns and Log Dividend Growth on Enterprise Value, IMA Equity, and CRSP Value Weighted Portfolio

Return	Time Period	Mean Return	Std Return	Std D growth
Enterprise Value	1946-2022	0.08	0.132	0.279
IMA Equity	1946-2022	0.082	0.15	0.15
CRSP VW	1946-2022	0.069	0.172	0.132

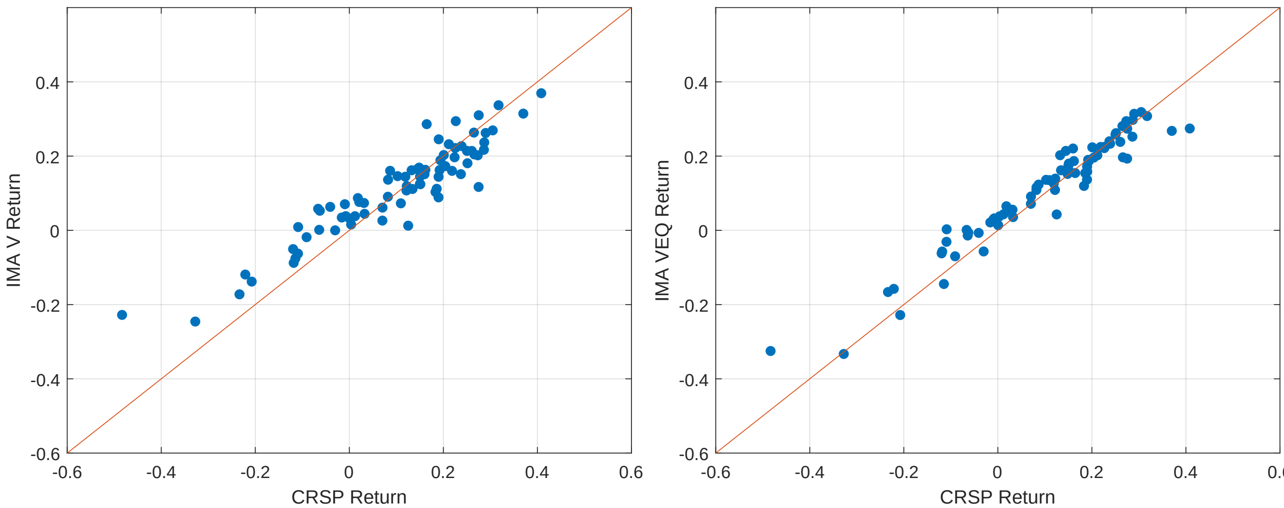

Figure B.2: Left Panel: Realized Returns on Enterprise Value vs. CRSP Value Weighted Return 1946-2023. Right Panel: Realized Returns on IMA Equity Value vs. CRSP Value Weighted Return: 1946-2022

C Appendix on Effective Tax Rates

Let r_{t+1}^{net} and r_{t+1}^{pretax} denote net real returns to capital between t and $t+1$ including taxes and before taxes:

$$\begin{aligned} r_{t+1}^{net} &= \frac{(1-\tau_{t+1}^c)}{(1-\lambda_t\tau_t^c)}\frac{R_{t+1}^K - Q_{t+1}\delta_{t+1}}{Q_t} + \frac{(1-\lambda_{t+1}\tau_{t+1}^c)}{(1-\lambda_t\tau_t^c)}\frac{Q_{t+1}}{Q_t} - 1, \\ r_{t+1}^{pretax} &= \frac{R_{t+1}^K - Q_{t+1}\delta_{t+1}}{Q_t} + \frac{Q_{t+1}}{Q_t} - 1. \end{aligned}$$

Define the effective tax rate on capital at t as the value for τ_t^e that satisfies

$$\mathbb{E}_t\left[M_{t,t+1}\left(1+(1-\tau_t^e)r_{t+1}^{gross}\right)\right] = \mathbb{E}_t\left[M_{t,t+1}\left(1+r_{t+1}^{net}\right)\right].$$

If we assume zero correlation between returns and the pricing kernel, we have

$$(1-\tau_t^e)\mathbb{E}_t\left[r_{t+1}^{pretax}\right] = \mathbb{E}_t\left[r_{t+1}^{net}\right]$$

and thus

$$\tau_t^e = \frac{\mathbb{E}_t\left[r_{t+1}^{pretax}\right] - \mathbb{E}_t\left[r_{t+1}^{net}\right]}{\mathbb{E}_t\left[r_{t+1}^{pretax}\right]},$$

which is the conventional way the marginal effective tax rate is defined (see, e.g., line 4 in Table 1 of Fullerton 1983).

In computing expected returns, we assume that all the tax parameters (τ_t^s, τ_t^c, τ_t^L and λ_t) are expected to remain unchanged between t and $t+1$.

Then

$$\mathbb{E}_t\left[r_{t+1}^{net}\right] = \frac{(1-\tau_t^c)}{(1-\lambda_t\tau_t^c)}\left(\frac{\mathbb{E}_t\left[\frac{R_{t+1}^K K_{t+1}}{GVA_{t+1}}\frac{GVA_{t+1}}{GVA_t}\right]}{\frac{Q_t K_{t+1}}{GVA_t}} - \mathbb{E}_t\left[\frac{Q_{t+1}}{Q_t}\delta_{t+1}\right]\right) + \mathbb{E}_t\left[\frac{Q_{t+1}}{Q_t}\right] - 1,$$

$$\mathbb{E}_t\left[r_{t+1}^{pretax}\right] = \left(\frac{\mathbb{E}_t\left[\frac{R_{t+1}^K K_{t+1}}{GVA_{t+1}}\frac{GVA_{t+1}}{GVA_t}\right]}{\frac{Q_t K_{t+1}}{GVA_t}} - \mathbb{E}_t\left[\frac{Q_{t+1}}{Q_t}\delta_{t+1}\right]\right) + \mathbb{E}_t\left[\frac{Q_{t+1}}{Q_t}\right] - 1.$$

Plugging these expressions into the formula for the effective tax rate gives

$$\tau_t^e = \frac{\left(\frac{\tau_t^c - \lambda_t\tau_t^c}{1-\lambda_t\tau_t^c}\right)}{1 + \frac{\mathbb{E}_t\left[\frac{Q_{t+1}}{Q_t}\right]-1}{\mathbb{E}_t\left[r_{t+1}^{pretax}\right]+1-\mathbb{E}_t\left[\frac{Q_{t+1}}{Q_t}\right]}}$$

When the expected gross growth rate for the relative price of investment is equal to one, this expression simplifies to

$$\tau_t^e = \frac{\tau_t^c - \lambda_t\tau_t^c}{1-\lambda_t\tau_t^c}.$$

Note that when $\lambda = 0$ (no expensing for net investment), the effective tax rate is equal to the statutory one: $\tau_t^e = \tau_t^c$.

When $\lambda = 1$ (full expensing for net investment), the effective tax is zero.

The series for τ_t^e plotted in Figure 7 is constructed assuming that $\mathbb{E}_t\left[\frac{Q_{t+1}}{Q_t}\right]$ is constant and equal to the average observed value over our sample period. The expected net pre-tax interest rate $\mathbb{E}_t\left[r_{t+1}^{pretax}\right]$ is computed as described in the text.

D Appendix on Value of Capital

The firm that owns and manages the physical capital stock takes as given an initial capital stock K_t and chooses future capital $\{K_{t+k}\}$ and after-tax free cash flow payable to owners $\{FCF_{t+k}^K\}$ for $k \geq 1$ to maximize

$$FCF_t^K + V_t^K$$

where

$$V_t^K = \sum_{k=1}^{\infty}\mathbb{E}_t\left[M_{t,t+k}FCF_{t+k}^K\right]$$

and

$$FCF_{t+k}^K = (1-\tau_{t+k}^c)\left(R_{t+k}^K - Q_{t+k}\delta_{t+k}\right)K_{t+k} - (1-\lambda_{t+k}\tau_{t+k}^c)Q_{t+k}\left(K_{t+k+1} - K_{t+k}\right).$$

The first-order condition with respect to K_{t+k+1} is

$$\mathbb{E}_{t+k}\left[M_{t,t+k+1}\left[(1-\tau_{t+k+1}^c)\left(R_{t+k+1}^K - Q_{t+k+1}\delta_{t+k+1}\right) + (1-\lambda_{t+k+1}\tau_{t+k+1}^c)Q_{t+k+1}\right]\right]$$
$$= M_{t,t+k}(1-\lambda_{t+k}\tau_{t+k}^c)Q_{t+k}. \quad (27)$$

Multiplying through by K_{t+k+1} gives

$$\mathbb{E}_{t+k}\left[M_{t,t+k+1}\left[(1-\tau^c_{t+k+1})\left(R^K_{t+k}-Q_{t+k+1}\delta_{t+k+1}\right)K_{t+k+1}+(1-\lambda_{t+k+1}\tau^c_{t+k+1})Q_{t+k+1}K_{t+k+1}\right]\right]$$
$$=M_{t,t+k}(1-\lambda_{t+k}\tau^c_{t+k})Q_{t+k}K_{t+k+1}. \tag{28}$$

The value of the firm managing the capital stock is

$$V^K_t=\sum_{k=1}^{\infty}\mathbb{E}_t\left[M_{t,t+k}\left[(1-\tau^c_{t+k})\left(R^K_{t+k}-Q_{t+k}\delta_{t+k}\right)K_{t+k}+(1-\lambda_{t+k}\tau^c_{t+k})Q_{t+k}K_{t+k}-(1-\lambda_{t+k}\tau^c_{t+k})Q_{t+k}K_{t+}\right.\right.$$

Using the equation (28), we can see that the term $-M_{t,t+k}(1-\lambda_{t+k}\tau^c_{t+k})Q_{t+k}K_{t+k+1}$ cancels with $\mathbb{E}_t\left[M_{t,t+k+1}\left[(1-\tau^c_{t+k+1})R^K_{t+k+1}K_{t+k+1}+(1-\lambda_{t+k+1}\tau^c_{t+k+1})Q_{t+k+1}(1-\delta_{t+k+1})K_{t+k+1}\right]\right]$ and so on moving up through time. The logic is that the value of free cash flow at $t+k$ that is sacrificed to increase next period capital must equal the expected value of the income plus resale value of that capital at $t+k+1$.

The only term that is left is

$$V^K_t=\mathbb{E}_t\left[M_{t,t+1}\left[(1-\tau^c_{t+1})\left(R^K_{t+1}-Q_{t+1}\delta_{t+1}\right)K_{t+1}+(1-\lambda_{t+1}\tau^c_{t+1})Q_{t+1}K_{t+1}\right]\right]$$

or, using equation (28) again, we have

$$V^K_t=(1-\lambda_t\tau^c_t)Q_tK_{t+1} \tag{29}$$

Note that if $\lambda_t=1$ (full expensing) then investment is not distorted, but firm value is depressed by $(1-\tau^c_t)$.

If $\lambda_t=0$, then investment and capital depressed when $\tau^c_t>0$, but the value of the firm is the replacement cost of its capital.

Note that one way to check that this is the right expression for valuation is to show that it gives the correct expression for the after-tax return to capital:

$$\begin{aligned}\frac{FCF^K_{t+1}+V^K_{t+1}}{V^K_t} &= \frac{(1-\tau^c_{t+1})\left(R^K_{t+1}-Q_{t+1}\delta_{t+1}\right)K_{t+1}-(1-\lambda_{t+1}\tau^c_{t+1})Q_{t+1}\left(K_{t+2}-K_{t+1}\right)+(1-\lambda_{t+1}\tau^c_{t+1})Q_{t+1}}{(1-\lambda_t\tau^c_t)Q_tK_{t+1}}\\ &= \frac{(1-\tau^c_{t+1})\left(R^K_{t+1}-Q_{t+1}\delta_{t+1}\right)K_{t+1}+(1-\lambda_{t+1}\tau^c_{t+1})Q_{t+1}K_{t+1}}{(1-\lambda_t\tau^c_t)Q_tK_{t+1}}\\ &= \frac{(1-\tau^c_{t+1})\left(\frac{R^K_{t+1}}{Q_t}-\frac{Q_{t+1}}{Q_t}\delta_{t+1}\right)+(1-\lambda_{t+1}\tau^c_{t+1})\frac{Q_{t+1}}{Q_t}}{(1-\lambda_t\tau^c_t)}\end{aligned}$$

which is the same as the return expression we derived earlier.

E Appendix on Pricing Kernel and Pricing Formulas

We start this section with proofs of Lemmas 1 and 2 and then present a general essentially affine model of the pricing kernel and a solution of that model. Here we will use Y as more compact notation for GVA.

E.1 Proofs of Lemmas 1 and 2:

Proof of Lemma 1: By definition, the ratio of the price of a claim to output one period ahead to current output is given by

$$\frac{P_t^{Y(1)}}{Y_t} = \mathbb{E}_t\left[M_{t,t+1}\frac{Y_{t+1}}{Y_t}\right].$$

Assume that this ratio is constant over time at a value $\beta = \frac{P^{Y(1)}}{Y}$. We then have

$$\frac{P_t^{Y(2)}}{Y_t} = \mathbb{E}_t\left[M_{t,t+1}\frac{Y_{t+1}}{Y_t}\frac{P_{t+1}^{Y(1)}}{Y_{t+1}}\right] = \beta\mathbb{E}_t\left[M_{t,t+1}\frac{Y_{t+1}}{Y_t}\right] = \beta^2,$$

where the first equality is by definition, and the second and third by the assumption that a price to a claim to output one period ahead relative to current output is constant over time. Iteration on this argument proves that

$$\frac{P_t^{Y(k)}}{Y_t} = \beta^k$$

which then proves the result.

Proof of Lemma 2: We have that a price to a claim to factorless income at horizon k satisfies the recursion

$$\frac{V_t^{\Pi(k+1)}}{Y_t} = \mathbb{E}_t\left[M_{t,t+1}\frac{Y_{t+1}}{Y_t}\frac{V_{t+1}^{\Pi(k)}}{Y_{t+1}}\right]$$

with

$$\frac{V_t^{\Pi(0)}}{Y_t} = \kappa_t$$

Applying this formula for $k = 1$ gives

$$\frac{V_t^{\Pi(1)}}{Y_t} = \mathbb{E}_t M_{t,t+1}\frac{Y_{t+1}}{Y_t}\kappa_{t+1} = \mathbb{E}_t M_{t,t+1}\frac{Y_{t+1}}{Y_t}\mathbb{E}_t\kappa_{t+1} + \mathbb{C}ov_t\left(M_{t,t+1}\frac{Y_{t+1}}{Y_t}, \kappa_{t+1} - \mathbb{E}_t\kappa_{t+1}\right) =$$

$$\mathbb{E}_t M_{t,t+1}\frac{Y_{t+1}}{Y_t}\mathbb{E}_t\kappa_{t+1} + \mathbb{C}ov_t\left(M_{t,t+1}\frac{Y_{t+1}}{Y_t}, \epsilon_{\kappa,t+1} + \epsilon_{x,t+1}\right) = \beta\mathbb{E}_t\kappa_{t+1} + C + D$$

where we have imposed our assumptions that both $\beta = P^{Y(1)}/Y$ and the conditional covariance in this expressions are constant over time. We then prove the lemma by induction. Assume the conditions of the lemma and

$$\frac{V_t^{\Pi(k)}}{Y_t} = \beta^k\mathbb{E}_t\kappa_{t+k} + \beta^{k-1}\left(\sum_{s=0}^{k-1}\rho^s C + kD\right)$$

Use the recursion to get

$$\begin{aligned}
\frac{V_t^{\Pi(k+1)}}{Y_t} &= \mathbb{E}_t M_{t+1} \frac{Y_{t+1}}{Y_t} \frac{V_{t+1}^{\Pi(k)}}{Y_{t+1}} \\
&= \mathbb{E}_t M_{t+1} \frac{Y_{t+1}}{Y_t} \left[\beta^k \mathbb{E}_{t+1} \kappa_{t+k+1} + \beta^{k-1} (\sum_{s=0}^{k-1} \rho^s C + kD) \right] \\
&= \beta^k \mathbb{E}_t M_{t+1} \frac{Y_{t+1}}{Y_t} \mathbb{E}_t \kappa_{t+k+1} + \beta^k \mathbb{C}ov_t \left(M_{t+1} \frac{Y_{t+1}}{Y_t}, \mathbb{E}_{t+1} \kappa_{t+k+1} - \mathbb{E}_t \kappa_{t+k+1} \right) \\
&\quad + \beta^{k-1} (\sum_{s=0}^{k-1} \rho^s C + kD) \mathbb{E}_t M_{t+1} \frac{Y_{t+1}}{Y_t} \\
&= \beta^{k+1} \mathbb{E}_t \kappa_{t+k+1} + \beta^k \mathbb{C}ov_t \left(M_{t+1} \frac{Y_{t+1}}{Y_t}, \rho^k \epsilon_{\kappa,t+1} + \epsilon_{x,t+1} \right) + \beta^k (\sum_{s=0}^{k-1} \rho^s C + kD) \\
&= \beta^{k+1} \mathbb{E}_t \kappa_{t+k+1} + \beta^k (\sum_{s=0}^{k} \rho^s C + (k+1) D)
\end{aligned}$$

which proves the result.

Note that we can derive equation (21) from

$$\frac{V_t^{\Pi}}{Y_t} = \sum_{k=1}^{\infty} \beta^k \left[\rho^k (\kappa_t - x_t) + x_t \right] + \beta^{k-1} \left(\left(\frac{1-\rho^k}{1-\rho} \right) C + kD \right)$$

E.2 Full Model of Pricing Kernel

We now turn to developing our full model of the pricing kernel that allows for time-varying risk premia. We use the notation in Section 8.3 page 247 of Campbell's textbook.

We start with a specification of the dynamics of the state variables and the pricing kernel. We then develop formulas for the prices of claims to zero coupon bond, zero coupon claims to aggregate output, and zero coupon claims to factorless income.

E.2.1 State Dynamics and Pricing Kernel

There is a column vector of state variables x_t, of length N. This vector includes observable outcomes and, potentially, unobserved latent states. All macroeconomic dynamics that we take as exogenous need to be in this list of state variables as well as any states we need for asset pricing.

This vector of state variables has dynamics given by

$$x_{t+1} = \Phi x_t + \Sigma \epsilon_{t+1} \tag{30}$$

Here Φ and Σ are $N \times N$ matrices, and ϵ_{t+1} is an $N \times 1$ vector of independent standard normal random variables.

For asset pricing, we are interested in the dynamics of two random variables. The first is the log SDF m_{t+1}, whose dynamics are given by

$$m_{t+1} = -(\delta_0 + \delta_1' x_t) - \frac{1}{2}\Lambda_t'\Lambda_t - \Lambda_t'\epsilon_{t+1} \tag{31}$$

Here δ_0 is a scalar, and δ_1 and Λ_t are $N \times 1$ vectors.

The vector Λ_t controls the conditional variance of the pricing kernel. This vector is given by

$$\Lambda_t = \Sigma^{-1}(\lambda_0 + \lambda_1 x_t) \tag{32}$$

where λ_0 is an $N \times 1$ vector and λ_1 is an $N \times N$ matrix.

We are interested in using this framework to price the following in our model. We wish to develop formulas for real interest rates (both short and long), for the value of a claim to real after-tax output of the corporate sector, for a claim to factorless income, and for the capital Euler equation governing the choice of end of period capital over output.

E.2.2 Pricing Zero Coupon Bonds

Our general framework offers a model for the prices of real zero coupon bonds as described in Campbell as follows. We price a claim at time t to one unit of consumption at time $t+k$. We denote this price as $P_t^{(k)}$ and solve for it from the recursion

$$P_t^{(k+1)} = \mathbb{E}_t \exp(m_{t+1}) P_{t+1}^{(k)}$$

starting from $P_t^{(0)} = 1$. The log of these bond prices, denoted by $p_t^{(k)} = \log(P_t^{(k)})$ has the form

$$p_t^{(k)} = A_k + B_k' x_t \tag{33}$$

with A_k a scalar and B_k and $N \times 1$ vector with $A_1 = -\delta_0$, $B_1 = -\delta_1$,

$$B_k' = B_{k-1}'(\Phi - \lambda_1) - \delta_1'$$

and

$$A_n = A_{n-1} - B_{n-1}'\lambda_0 - \delta_0 + \frac{1}{2}B_{n-1}'\Sigma\Sigma' B_{n-1}$$

These expressions are presented in Campbell and can be derived along the lines of how we derive the price of a claim to output below.

E.2.3 Pricing a Claim to Output

We now turn to pricing a claim to output. We denote the price of a claim at t to aggregate after-tax output at $t+k$ by $P_t^{Y(k)}$. Note that these prices satisfy the recursion

$$\frac{P_t^{Y(k+1)}}{Y_t} = \mathbb{E}_t \exp(m_{t+1}) \frac{Y_{t+1}}{Y_t}\frac{P_{t+1}^{Y(k)}}{Y_{t+1}} = \mathbb{E}_t \exp(m_{t+1} + g_{Y,t+1}) \frac{P_{t+1}^{Y(k)}}{Y_{t+1}}$$

This recursion looks the same as that for zero coupon bonds, but now we have the price to output ratio $\frac{P_t^{Y(k+1)}}{Y_t}$ instead of the zero coupon bond price $P_t^{(k+1)}$ and we have the sum of the log SDF and output growth $m_{t+1} + g_{Y,t+1}$ instead of simply m_{t+1}. Thus, we look for a solution of the form

$$pd_t^{Y(k)} \equiv \log\left(\frac{P_t^{Y(k)}}{Y_t}\right) = C_k + D_k' x_t$$

To derive this solution, we specifically assume that the dynamics of output growth are given by

$$g_{Y,t+1} = \gamma_0 + \gamma_1' x_t + \eta_g' \epsilon_{t+1} - \frac{1}{2}\eta_g' \eta_g \tag{34}$$

where γ_0 is mean log output growth, γ_1 is a vector indicating how expected output growth varies with the state, and η_g' corresponds to $\gamma_1' \Sigma$ relevant for impulses to output growth.

Lemma: The solution for these prices is given by

$$\frac{P_t^{Y(k)}}{Y_t} = \exp\left(C_k + D_k' x_t\right) \tag{35}$$

with $C_0 = 0$, $D_0' = 0$,

$$D_k' = -(\delta_1' - \gamma_1') - \eta_g' \Sigma^{-1} \lambda_1 + D_{k-1}'(\Phi - \lambda_1)$$

and

$$C_k = C_{k-1} - (\delta_0 - \gamma_0) - \eta_g' \Sigma^{-1} \lambda_0 - D_{k-1}' \lambda_0 + \frac{1}{2} D_{k-1}' \Sigma \Sigma' D_{k-1}$$

We derive this formula in section E.4.

Note that the expected return on a claim to output one period ahead is given by

$$\frac{Y_t}{P_t^{Y(1)}} \mathbb{E}_t \exp(g_{Y,t+1}) = \exp(-C_1 - D_1' x_t + \gamma_0 + \gamma_1' x_t) =$$

$$\exp\left(\delta_0 + \delta_1' x_t + \eta_g' \Sigma^{-1}(\lambda_0 + \lambda_1 x_t)\right)$$

Important Lemma: If

$$0 = (\delta_1' - \gamma_1') + \eta_g' \Sigma^{-1} \lambda_1 \tag{36}$$

then the price dividend ratio for a claim to output is constant no matter what else is going on with asset prices.

Proof: From the recursion for D_k, since we start with $D_0 = 0$, these conditions imply that $D_k = 0$ for all k. This then implies that

$$\frac{P_t^{Y(k)}}{Y_t} = \exp\left(C_k\right) \tag{37}$$

which is constant. Note that in this case

$$C_k = k\left[-(\delta_0 - \gamma_0) - \eta_g' \Sigma^{-1} \lambda_0\right]$$

and the ratio of the value of a claim to output relative to output is given by a constant

$$\frac{P_t^Y}{Y_t} = \frac{1}{1-\exp\left(C_1\right)}$$

Interpretation What does the condition

$$0 = (\delta_1' - \gamma_1') + \eta_g'\Sigma^{-1}\lambda_1$$

mean? The vector δ_1' governs how the risk free interest rate moves with the state x_t and the vector γ_1' governs how the conditional expectation of the growth of log after tax output moves with the state x_t. Thus, $\delta_1' - \gamma_1'$ governs how $r - g$ relevant for pricing a claim to output one period ahead moves with the state x_t. The vector $\eta_g'\Sigma^{-1}\lambda_1 = 0$ governs how the risk correction to a claim to output one period ahead moves with the state x_t. Thus, the condition of the lemma is a condition that all observed movements in the gap between the risk free interest rate and expected growth of after tax output are movements in the risk adjustment on a claim to after tax output one period ahead. This is a general version of the assumptions we made in the baseline case in the prior set of notes. If one were to estimate this model, we believe that this is a restriction on parameters that one could impose on the estimation and then check whether it is rejected or not.

E.2.4 Pricing claims to factorless income

Now we turn to pricing a claim to factorless income. The price at t to a claim to factorless income at $t+k+1$ satisfies

$$\frac{V_t^{\Pi(k+1)}}{Y_t} = \mathbb{E}_t \exp(\sum_{s=0}^{k} m_{t+s+1} + g_{Y,t+s+1})\kappa_{t+k+1} =$$

$$\mathbb{E}_t \exp(\sum_{s=0}^{k} m_{t+s+1} + g_{Y,t+s+1})\mathbb{E}_t\kappa_{t+k+1} + \mathbb{C}ov_t\left(\exp(\sum_{s=0}^{k} m_{t+s+1} + g_{Y,t+s+1}), \kappa_{t+k+1}\right)$$

This argument gives us a key pricing relationship for a claim to factorless income at horizon $k+1$

$$\frac{V_t^{\Pi(k+1)}}{Y_t} = \frac{P_t^{Y(k+1)}}{Y_t}\mathbb{E}_t\kappa_{t+k+1} + \mathbb{C}ov_t\left(\exp(\sum_{s=0}^{k} m_{t+s+1} + g_{Y,t+s+1}), \kappa_{t+k+1}\right) \tag{38}$$

The value of a claim to factorless income at t is then given by the sum of these terms across horizons k

$$\frac{V_t^{\Pi}}{Y_t} = \sum_{k=1}^{\infty} \frac{P_t^{Y(k+1)}}{Y_t}\mathbb{E}_t\kappa_{t+k+1} + \mathbb{C}ov_t\left(\exp(\sum_{s=0}^{k} m_{t+s+1} + g_{Y,t+s+1}), \kappa_{t+k+1}\right)$$

Thus, the price of a claim to factorless income at horizon $k+1$ relative to output can move for three reasons. First, the price of a claim to output at horizon $k+1$ relative to output at t given by $\frac{P_t^{Y(k+1)}}{Y_t}$ might move. Second, the expected factorless income share might move. And third, the risk premium in the covariance term might move.

We look to develop an analytical solution for the price to a claim to factorless income by making use of the following formula. If x and y and z are independent standard normal random variables and a, b, c, d are scalar constants, then

$$\mathbb{E}\exp(ax+by)(cx+dz) = ca\exp((a^2+b^2)/2) \tag{39}$$

We derive this formula in section E.5. It is a special case of Stein's Lemma. Note that since x, y, and z all have mean zero, $\mathbb{E}\exp(ax+by)\mathbb{E}(cx+dz) = 0$ and hence

$$\mathbb{C}ov\left(\exp(ax+by), cx+dz\right) = ca\exp((a^2+b^2)/2)$$

Also note that if ϵ is an $N\times 1$ vector of independent standard normal random variables and A and B are $N\times 1$ vectors, then

$$\mathbb{E}\exp(A'\epsilon)B'\epsilon = \exp\left(\frac{1}{2}A'A\right)A'B$$

We use this version of this formula below.

We proceed as follows. We assume that the share of factorless income in after tax output κ_t is one of the elements of the state vector and x_t is another element of this vector. Assume that μ and the matrices Φ and Σ are consistent with

$$\kappa_{t+1} = \kappa_t + x_t + \eta_\kappa'\epsilon_{t+1} \tag{40}$$

$$x_{t+1} = \rho x_t + \theta_x'\epsilon_{t+1} \tag{41}$$

These equations imply that

$$\mathbb{E}_t\kappa_{t+k+1} = \kappa_t + \left(\frac{1-\rho^k}{1-\rho}\right)x_t$$

and

$$\mathbb{E}_t\kappa_{t+\infty} - \mathbb{E}_t\kappa_{t+k+1} = \rho^k\frac{x_t}{1-\rho}$$

Thus, we can consider the gap between the expected share of factorless income from next period on to converge to its long run value at a rate ρ with the initial gap being $\frac{x_t}{1-\rho}$

To solve for the value of factorless income, we have the recursion

$$\frac{V_t^{\Pi(k+1)}}{Y_t} = \mathbb{E}_t\exp(m_{t+1}+g_{Y,t+1})\frac{V_{t+1}^{\Pi(k)}}{Y_{t+1}}$$

with

$$\frac{V_t^{\Pi(0)}}{Y_t} = \kappa_t$$

Lemma: The price for a claim to factorless income at horizon k is given by

$$\frac{V_t^{\Pi(k)}}{Y_t} = \frac{P_t^{Y(k)}}{Y_t} \left(\mathbb{E}_t \kappa_{t+k} + F_k + G_k' x_t \right) \tag{42}$$

with $F_0 = 0$ and $G_0' = 0$ and

$$G_k' = G_{k-1}' \left(\Phi - \lambda_1 \right) - (\eta_\kappa' + \theta_{x\kappa}') \Sigma^{-1} \lambda_1$$

and

$$F_k = F_{k-1} + G_{k-1}' \mu + \left(\eta_\kappa' + \theta_{x\kappa}' + G_{k-1}' \Sigma \right) \left(\Sigma' D_{k-1} - \Sigma^{-1} \lambda_0 + \eta_g \right)$$

Proof: In subsection E.6

Important Special Case: Assume condition 36 holds so that $D_k = 0$ for all k the prices of claims to output at different horizons are all constant at those given by equation (37). Assume as well that $(\eta_\kappa' + \theta_{x\kappa}') \Sigma^{-1} \lambda_1 = 0$ so that the risk on factorless income share shocks is independent of the state vector x_t. Then $G_k' = 0$ for all k and thus

$$F_k = k \left(\eta_\kappa' + \theta_{x\kappa}' \right) \left(-\Sigma^{-1} \lambda_0 + \eta_g \right) = k F_1$$

In this case, we have

$$\frac{V_t^{\Pi}}{Y_t} = \frac{1}{1 - \exp\left(C_1 \right)} \left[\kappa_t + \frac{x_t}{1 - \rho} \right] - \frac{1}{1 - \rho \exp\left(C_1 \right)} \frac{x_t}{1 - \rho} + F_1 \frac{\exp(C_1)}{(1 - \exp(C_1))^2}$$

E.3 The Euler Equation for Capital

The Euler equation for capital plays a key role in our model. To solve the Euler equation, we assume the dynamics of the capital price Q_t are given by

$$\log(Q_{t+1}) - \log(Q_t) = \zeta_0 + \zeta_1' x_t + \eta_Q' \epsilon_{t+1} - \frac{1}{2} \eta_Q' \eta_Q$$

where ζ_0 is mean growth of Q_t, ζ_1 is an $N \times 1$ vector indicating how expected growth of Q_t varies with the state, and η_Q' corresponds to $\zeta_1' \Sigma$ relevant for impulses to the growth of the capital price.

This is given by

Lemma: Capital Output Ratio

$$\frac{Q_t K_{t+1}}{Y_t} = \tag{43}$$

$$\alpha \frac{\left(\frac{P_t^{Y(1)}}{Y_t} - \frac{V_t^{\Pi(1)}}{Y_t} \right)}{1 - (1 - \delta_{t+1}) \exp(H_1 + J_1' x_t)}$$

with

$$J_1' = -(\delta_1' - \zeta_1') - \eta_Q' \Sigma^{-1} \lambda_1$$

and

$$H_1 = -(\delta_0 - \zeta_0) - \eta_Q' \Sigma^{-1} \lambda_0$$

where we have used the formulas for the price of a claim to output and to factorless income one period ahead in our Lemmas above.

Proof: See Subsection E.7

E.4 Proof of Formula (35)

We have the dynamics of output growth given by

$$g_{Y,t+1} = \gamma_0 + \gamma_1' x_t + \eta_g' \epsilon_{t+1} - \frac{1}{2} \eta_g' \eta_g$$

where γ_0 is mean output growth, γ_1 is a vector indicating how expected output growth varies with the state, and η_g' corresponds to $\gamma_1' \Sigma$ relevant for impulses to output growth.

From equation (31), we can write

$$m_{t+1} + g_{Y,t+1} = -(\delta_0 - \gamma_0 + (\delta_1' - \gamma_1')x_t) - \frac{1}{2}\Lambda_t'\Lambda_t - (\Lambda_t' - \eta_g')\epsilon_{t+1} - \frac{1}{2}\eta_g'\eta_g \tag{44}$$

The recursion that we need to solve for C_k and D_k is

$$\exp(C_k + D_k' x_t) =$$

$$\mathbb{E}_t \exp\left(-(\delta_0 - \gamma_0 + (\delta_1' - \gamma_1')x_t) - \frac{1}{2}\Lambda_t'\Lambda_t - (\Lambda_t' - \eta_g')\epsilon_{t+1} - \frac{1}{2}\eta_g'\eta_g + C_{k-1} + D_{k-1}'(\mu + \Phi x_t + \Sigma\epsilon_{t+1})\right)$$

Note that the conditional variance of this term in parentheses is given by

$$(\Lambda_t' - \eta_g' - D_{k-1}'\Sigma)(\Lambda_t - \eta_g - \Sigma' D_{k-1}) = \Lambda_t'\Lambda_t - 2\eta_g'\Lambda_t - 2D_{k-1}'\Sigma\Lambda_t + \eta_g'\eta_g + D_{k-1}'\Sigma\Sigma' D_{k-1}$$

Using the standard expectation of a log normal random variable formula to compute the expectation, we then get the recursion that

$$C_k + D_k' x_t = -(\delta_0 - \gamma_0 + (\delta_1' - \gamma_1')x_t) + C_{k-1} + D_{k-1}'(\mu + \Phi x_t - \lambda_0 - \lambda_1 x_t) - \eta_g' \Sigma^{-1}(\lambda_0 + \lambda_1 x_t)$$

$$+\frac{1}{2} D_{k-1}' \Sigma\Sigma' D_{k-1}$$

Matching terms on x_t gives the recursion for D_k' and then matching constants gives the recursion for C_k.

E.5 Proof of Formula (39)

One can prove this formula using the moment generating function for normal random variables. In particular, we start by computing for a normal random variable

$$\mathbb{E}\exp(atx) = \exp(at\mu + \frac{1}{2}a^2t^2\sigma^2)$$

We then have

$$\mathbb{E}ax\exp(atx) = \mathbb{E}\frac{d}{dt}\exp(atx) = \exp(at\mu + \frac{1}{2}a^2t^2\sigma^2)(a\mu + ta^2\sigma^2)$$

If we evaluate this expression at $t = 1$ with $\mu = 0$ and $\sigma = 1$ for a standard normal, we have

$$\mathbb{E}ax\exp(ax) = \exp(\frac{1}{2}a^2)a^2$$

we multiply by c/a to obtain

$$\mathbb{E}cx\exp(ax) = \exp(\frac{1}{2}a^2)ca$$

We then have

$$\mathbb{E}\exp(ax + by)(cx + dz) = \mathbb{E}\exp(by)\mathbb{E}cx\exp(ax) + \mathbb{E}\exp(by)\mathbb{E}\exp(ax)\mathbb{E}dz$$

by the independence of x, y and z. Finally, since $\mathbb{E}z = 0$ and $\mathbb{E}\exp(by) = \exp(\frac{1}{2}b^2)$ we get equation (39).

E.6 Proof of Formula 42

We solve for the coefficients F_k and G'_k as follows.

We have

$$\frac{P^{(k)}_{Yt}}{Y_t}\left(\mathbb{E}_t\kappa_{t+k} + F_k + G'_k x_t\right) =$$

$$\mathbb{E}_t\exp(m_{t+1} + g_{Y,t+1})\frac{P^{(k-1)}_{Yt+1}}{Y_{t+1}}\left(\mathbb{E}_{t+1}\kappa_{t+k} + F_{k-1} + G'_{k-1}x_{t+1}\right) =$$

$$\mathbb{E}_t\exp(m_{t+1} + g_{Y,t+1})\frac{P^{(k-1)}_{Yt+1}}{Y_{t+1}}\left(\mathbb{E}_t\kappa_{t+k} + F_{k-1} + G'_{k-1}\mathbb{E}_t x_{t+1}\right) +$$

$$\mathbb{E}_t\exp(m_{t+1} + g_{Y,t+1})\frac{P^{(k-1)}_{Yt+1}}{Y_{t+1}}\left(\mathbb{E}_{t+1}\kappa_{t+k} - \mathbb{E}_t\kappa_{t+k} + G'_{k-1}\left(x_{t+1} - \mathbb{E}_t x_{t+1}\right)\right) =$$

$$\frac{P^{(k-1)}_{Yt}}{Y_t}\left(\mathbb{E}_t\kappa_{t+k} + F_{k-1} + G'_{k-1}\left(\mu + \Phi x_t\right)\right) +$$

$$\mathbb{E}_t\exp(m_{t+1} + g_{Y,t+1})\frac{P^{(k)}_{Yt+1}}{Y_{t+1}}\left((\eta'_\kappa + \theta'_{x\kappa})\epsilon_{t+1} + G'_{k-1}\Sigma\epsilon_{t+1}\right)$$

where we have use the results that

$$\mathbb{E}_{t+1}\kappa_{t+k} - \mathbb{E}_t\kappa_{t+k} = (\eta'_\kappa + \theta'_{x\kappa})\epsilon_{t+1}$$

and

$$x_{t+1} - \mathbb{E}_t x_{t+1} = \Sigma\epsilon_{t+1}$$

We then can expand terms to get

$$\frac{P_{Yt}^{(k)}}{Y_t}\left(F_k + G_k' x_t\right) =$$

$$\frac{P_{Yt}^{(k)}}{Y_t}\left(F_{k-1} + G_{k-1}'\left(\mu + \Phi x_t\right)\right) +$$

$$\mathbb{E}_t \exp\left(-(\delta_0 - \gamma_0 + (\delta_1' - \gamma_1')x_t) - \frac{1}{2}\Lambda_t'\Lambda_t - (\Lambda_t' - \eta_g')\epsilon_{t+1} + C_{k-1} + D_{k-1}'(\mu + \Phi x_t + \Sigma\epsilon_{t+1})\right) \times$$

$$\left(\eta_\kappa' + \theta_{x\kappa}' + G_{k-1}'\Sigma\right)\epsilon_{t+1} =$$

$$\frac{P_{Yt}^{(k)}}{Y_t}\left(F_{k-1} + G_{k-1}'\left(\mu + \Phi x_t\right)\right) +$$

$$\frac{P_{Yt}^{(k)}}{Y_t}\left(\eta_\kappa' + \theta_{x\kappa}' + G_{k-1}'\Sigma\right)\left(\Sigma' D_{k-1} - \Sigma^{-1}\left(\lambda_0 + \lambda_1 x_t\right) + \eta_g\right)$$

We can use this equality to get formulas for the coefficients G_k and F_k. Matching terms on x_t gives

$$G_k' = G_{k-1}'\left(\Phi - \lambda_1\right) - (\eta_\kappa' + \theta_{x\kappa}')\Sigma^{-1}\lambda_1$$

Matching constant terms gives

$$F_k = F_{k-1} + G_{k-1}'\mu + \left(\eta_\kappa' + \theta_{x\kappa}' + G_{k-1}'\Sigma\right)\left(\Sigma' D_{k-1} - \Sigma^{-1}\lambda_0 + \eta_g\right)$$

E.7 Solution of the Capital Euler Equation:

To prove this result, note that the capital Euler equation can be written as

$$1 = \mathbb{E}_t \exp(m_{t+1})\frac{Y_{t+1}}{Y_t}\alpha(1 - \kappa_{t+1})\frac{Y_t}{Q_t K_{t+1}} + \mathbb{E}_t \exp(m_{t+1})(1 - \delta_{t+1})\frac{Q_{t+1}}{Q_t}$$

Note that this equation can be written as

$$\frac{Q_t K_{t+1}}{Y_t} = \alpha\frac{\left(\frac{P_t^{Y(1)}}{Y_t} - \frac{V_t^{\Pi(1)}}{Y_t}\right)}{1 - \mathbb{E}_t \exp(m_{t+1})(1 - \delta_{t+1})\frac{Q_{t+1}}{Q_t}}$$

Note also that, given our timing assumption regarding the realization of the depreciation rate and the dynamics of Q_t, the term

$$\mathbb{E}_t \exp(m_{t+1})(1 - \delta_{t+1})\frac{Q_{t+1}}{Q_t} = (1 - \delta_{t+1})\mathbb{E}_t \exp(m_{t+1} + g_{Q,t+1})$$

We then have from our assumed dynamics for capital that

$$\mathbb{E}_t \exp(m_{t+1} + g_{Qt+1}) = \exp(H_1 + J_1' x_t)$$

with

$$J_1' = -(\delta_1' - \zeta_1') - \eta_Q' \Sigma^{-1} \lambda_1$$

and

$$H_1 = -(\delta_0 - \zeta_0) - \eta_Q' \Sigma^{-1} \lambda_0$$

F Comparison to Shiller (1981)

Here we compare an old-style Campbell-Shiller excess volatility test with what we do on our paper.

We assume that in the data, we have a time series for the realized share of factorless income denoted by κ_t^D where the superscript D denotes the data and the dates span from $t = 0$ to $T + 1$. We assume that we have data on the ratio of the value of factorless income to output denoted by

$$\frac{V_t^{\Pi,D}}{Y_t}$$

where again the superscript D refers to data.

We model the value of factorless income as

$$\frac{V_t^{\Pi,\star}}{Y_t} = \sum_{k=1}^{\infty} \beta^k \mathbb{E}_t \kappa_{t+k}^{\star} + \phi$$

where β is a parameter that we choose related to the price dividend ratio for a claim to output, ϕ is a constant risk adjustment, and $\mathbb{E}_t \kappa_{t+k}^{\star}$ is a model of the value of κ_{t+k} expected at t that we impose in our excess volatility calculation.

Given this valuation model, we have innovations to valuation from period t to $t+1$ given by

$$\begin{aligned}
\frac{V_{t+1}^{\Pi,\star}}{Y_{t+1}} - \frac{V_t^{\Pi,\star}}{Y_t} &= \sum_{k=1}^{\infty} \beta^k \mathbb{E}_{t+1} \kappa_{t+1+k}^{\star} - \beta \sum_{k=1}^{\infty} \beta^k \mathbb{E}_t \kappa_{t+1+k}^{\star} - \beta \mathbb{E}_t \kappa_{t+1}^{\star} \\
&= \sum_{k=1}^{\infty} \beta^k \left[\mathbb{E}_{t+1} \kappa_{t+1+k}^{\star} - \beta \mathbb{E}_t \kappa_{t+1+k}^{\star} \right] - \beta \mathbb{E}_t \kappa_{t+1}^{\star} \\
&= \sum_{k=1}^{\infty} \beta^k \left[\mathbb{E}_{t+1} \kappa_{t+1+k}^{\star} - \mathbb{E}_t \kappa_{t+1+k}^{\star} + (1-\beta) \mathbb{E}_t \kappa_{t+1+k}^{\star} \right] - \beta \mathbb{E}_t \kappa_{t+1}^{\star} \\
&= \sum_{k=1}^{\infty} \beta^k \left[\mathbb{E}_{t+1} \kappa_{t+1+k}^{\star} - \mathbb{E}_t \kappa_{t+1+k}^{\star} \right] + (1-\beta) \sum_{k=0}^{\infty} \beta^k \mathbb{E}_t \kappa_{t+1+k}^{\star} - \mathbb{E}_t \kappa_{t+1}^{\star}
\end{aligned}$$

To summarize, innovations to the value of factorless income are given by

$$\frac{V_{t+1}^{\Pi,\star}}{Y_{t+1}} - \frac{V_t^{\Pi,\star}}{Y_t} = \sum_{k=1}^{\infty} \beta^k \left[\mathbb{E}_{t+1} \kappa_{t+1+k}^{\star} - \mathbb{E}_t \kappa_{t+1+k}^{\star} \right] + (1-\beta) \sum_{k=0}^{\infty} \beta^k \mathbb{E}_t \kappa_{t+1+k}^{\star} - \mathbb{E}_t \kappa_{t+1}^{\star} \tag{45}$$

where

$$\sum_{k=1}^{\infty} \beta^k \left[\mathbb{E}_{t+1}\kappa^{\star}_{t+1+k} - \mathbb{E}_t \kappa^{\star}_{t+1+k} \right]$$

is a weighted sum of innovations to the expected value of future κ_{t+1+k} implied by a model of expectations, and

$$(1-\beta) \sum_{k=0}^{\infty} \beta^k \mathbb{E}_t \kappa^{\star}_{t+1+k} - \mathbb{E}_t \kappa^{\star}_{t+1}$$

is a comparison of a weighted average of expectations of future κ_{t+k+1} and the expectation at t of κ_{t+1}.

F.1 What Shiller (1981) Did

Shiller used realized values of dividends in his calculation. Specifically, let us set

$$\mathbb{E}_t \kappa^{\star}_{t+1+k} = \kappa^D_{t+1+k}$$

for $k \geq 1$. This implies that the terms

$$\mathbb{E}_{t+1}\kappa^{\star}_{t+1+k} - \mathbb{E}_t \kappa^{\star}_{t+1+k} = 0$$

and we are left with model-implied volatility of valuations as

$$\frac{V^{\Pi,\star}_{t+1}}{Y_{t+1}} - \frac{V^{\Pi,\star}_{t}}{Y_t} = (1-\beta) \sum_{k=0}^{\infty} \beta^k \kappa^D_{t+1+k} - \kappa^D_{t+1}$$

F.2 What We Do

We assume a model for expectations based on

$$\kappa_{t+1} - x^{\star}_{t+1} = \rho(\kappa_t - x^{\star}_t) + \epsilon_{\kappa,t+1} \tag{46}$$

$$x^{\star}_{t+1} = x^{\star}_t + \epsilon_{x,t+1} \tag{47}$$

where $x^{\star}_t$ is an unobserved variable that we choose as part of the model. We assume that κ^D_t is taken from the data. This implies that we compute innovations

$$\epsilon_{x,t+1} = x^{\star}_{t+1} - x^{\star}_t$$

from the model and then compute innovations

$$\epsilon_{\kappa,t+1} = \kappa^D_{t+1} - x^{\star}_{t+1} - \rho(\kappa^D_t - x^{\star}_t)$$

from the data given the model specification of $x^{\star}_t$ and ρ.

These equations imply that

$$\mathbb{E}_t \kappa^{\star}_{t+k} = \rho^k (\kappa^D_t - x^{\star}_t) + x^{\star}_t$$

and

$$\mathbb{E}_t \kappa^\star_{t+\infty} = x^\star_t$$

With these dynamics of the factorless income share, we have for $k \geq 0$, innovations to expectations of the factorless income share at horizon k are given by

$$\mathbb{E}_{t+1}\kappa^\star_{t+k+1} - \mathbb{E}_t\kappa^\star_{t+k+1} = \rho^k(\kappa^D_{t+1} - x^\star_{t+1}) - \rho^{k+1}(\kappa^D_t - x^\star_t) + x^\star_{t+1} - x^\star_t = \rho^k\epsilon_{\kappa,t+1} + \epsilon_{x,t+1}$$

Using these results with our equation (45) gives us that our model implied changes in valuation are given by

$$\begin{aligned}
\frac{V^{\Pi,\star}_{t+1}}{Y_{t+1}} - \frac{V^{\Pi,\star}_t}{Y_t} &= \sum_{k=1}^{\infty}\beta^k\left[\rho^k\epsilon_{\kappa,t+1} + \epsilon_{x,t+1}\right] + (1-\beta)\sum_{k=0}^{\infty}\beta^k\rho^{k+1}(\kappa^D_t - x^\star_t) - \rho(\kappa^D_t - x^\star_t) \\
&= \frac{\rho\beta}{1-\rho\beta}\epsilon_{\kappa,t+1} + \frac{\beta}{1-\beta}\epsilon_{x,t+1} - \left(\frac{(1-\rho)\beta}{1-\rho\beta}\right)\rho(\kappa^D_t - x^\star_t) \\
&= \frac{\rho\beta}{1-\rho\beta}\left(\kappa^D_{t+1} - \kappa^D_t\right) + \left(\frac{\beta}{1-\beta} - \frac{\rho\beta}{1-\rho\beta}\right)\left(x^\star_{t+1} - x^\star_t\right)
\end{aligned}$$

We can draw a number of conclusions from this formula.

First consider the case in which we assume that $x^\star_t$ is fixed over time. Then the term $x^\star_{t+1} - x^\star_t = 0$. Then the volatility of values implied by the model are pinned down from parameter choices ρ, β, and the data on $\kappa^D_{t+1} - \kappa^D_t$.

Second, if we add variation over time in $x^\star_{t+1} - x^\star_t$, then this adds to model-implied volatility if $\rho < 1$. If $\rho = 1$, then the term connected to $x^\star_{t+1} - x^\star_t$ disappears.

Third, one can interpret what we are doing as follows. We have

$$\frac{V^{\Pi,\star}_{t+1}}{Y_{t+1}} - \frac{V^{\Pi,\star}_t}{Y_t}$$

and

$$\kappa^D_{t+1} - \kappa^D_t$$

from the data. Then, given a choice of parameters ρ and β we are solving for $x^\star_{t+1} - x^\star_t$ to rationalize these data. The required variation in $x^\star_t$ can be small if the term $\left(\frac{\beta}{1-\beta} - \frac{\rho\beta}{1-\rho\beta}\right)$ is large.