Notes on Mirrlees Taxation

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1 Problem Setup

Static problem:

Agents differ by productivity θ

I values for productivity $\theta_1, ..., \theta_I$

Fraction π_i of each type

Preferences

$$\begin{aligned} U_i &= u(c_i) - v\left(\frac{y_i}{\theta_i}\right) \\ &= \log(c_i) - \frac{\left(\frac{y_i}{\theta_i}\right)^{1+\sigma}}{1+\sigma} \end{aligned}$$

Planner must raise revenue to finance G

Planner puts weight W_i on type i s.t. $\sum_i W_i \pi_i = 1$

An allocation is a vector $\{(c_i, y_i)\}_{i=1}^{I}$

Social welfare is given by

$$\sum_i W_i \pi_i \left\{u(c_i) - v\left(\frac{y_i}{\theta_i}\right)\right\}$$

Planner can observe y, but not θ

So taxes must be a function of y

Planner's problem is therefore to choose a tax function $T(y)$ such that when agents take this schedule as given and solve

$$\max_{\{c_i, y_i\}} \left\{u(c_i) - v\left(\frac{y_i}{\theta_i}\right)\right\}$$

$$\text{s.t. } c_i = y_i - T(y_i)$$

the resulting allocations maximize social welfare.

Model could be interpred as model in which all agents have same wage, but different disutilities of work. Thus, interpretations in which workers have different labor productivity but identical preferences versus interpretation in which they are equally productive but have different disutility of work are formally identical. The idea is that in terms of individual choices it doesn't matter whether it is costly for a worker to deliver output because she must work a lot of hours or because she just hates working. (Though a planner might feel differently about these two people in terms of Pareto weights)

2 Mirrlees' Clever Idea

Now the problem is that the optimal T could be a very complicated non-parametric function. How are we supposed to solve for it?

Mirrlees' clever idea.

Instead of thinking of planner picking T think of planner picking allocations directly.

In particular think of planner as offering a menu of different choices $\{(c_i, y_i)\}$ with one pair in this menu intended for each type. The planner can say:

"If you produce income y_i (which I can observe) then you must pay a tax $y_i - c_i$."

But the planner cannot force agents to choose the pair intended for their type, because type is not observed

Thus the planner must incentivize choosing the appropriate allocation by making sure that each type weakly prefers to pick their intended allocation

Thus the Mirrlees problem is

$$
\begin{aligned}
& \max_{\{c_i, y_i\}} \sum_i W_i \pi_i \left\{ u(c_i) - v\left(\frac{y_i}{\theta_i}\right) \right\} \\
& s.t. \\
u(c_i) - v\left(\frac{y_i}{\theta_i}\right) &\geq u(c_j) - v\left(\frac{y_j}{\theta_i}\right) \text{ for all } i, j \\
\sum_i \pi_i c_i + G &= \sum_i \pi_i y_i
\end{aligned}
$$

There are lots of incentive constraints!

Fortunately most of them will not be binding

Suppose planner wants to redistribute downwards => lower taxes on less productive agents => possible incentive to pretend to be less productive, but no incentive to pretend to be more productive => only downward IC constraints will bind.

In fact, only local downward constraints will bind.

So we can simplify the problem to

$$\max_{\{c_i,y_i\}} \sum_i W_i \pi_i \left\{ u(c_i) - v\left(\frac{y_i}{\theta_i}\right) \right\}$$

$$\pi_i \mu_i \quad : \quad u(c_i) - v\left(\frac{y_i}{\theta_i}\right) \geq u(c_{i-1}) - v\left(\frac{y_{i-1}}{\theta_i}\right) \text{ for } i = 2, ..., I$$

$$\lambda \quad : \quad \sum_i \pi_i c_i + G \leq \sum_i \pi_i y_i$$

FOCs (recall no IC constraint for $i = 1 => \mu_1 = 0$)

$$c_i \quad : \quad W_i \pi_i u'(c_i) + \pi_i \mu_i u'(c_i) - \lambda \pi_i - \pi_{i+1} \mu_{i+1} u'(c_i) = 0 \text{ for } i = 1, ..., I-1$$

$$y_i \quad : \quad -W_i \pi_i v'\left(\frac{y_i}{\theta_i}\right) \frac{1}{\theta_i} - \pi_i \mu_i v'\left(\frac{y_i}{\theta_i}\right) \frac{1}{\theta_i} + \lambda \pi_i + \pi_{i+1} \mu_{i+1} v'\left(\frac{y_i}{\theta_{i+1}}\right) \frac{1}{\theta_{i+1}} = 0 \text{ for } i = 1, ..., I-1$$

$$c_i \quad : \quad W_i \pi_i u'(c_i) + \pi_i \mu_i u'(c_i) - \lambda \pi_i = 0 \text{ for } i = I$$

$$y_i \quad : \quad -W_i \pi_i v'\left(\frac{y_i}{\theta_i}\right) \frac{1}{\theta_i} - \pi_i \mu_i v'\left(\frac{y_i}{\theta_i}\right) \frac{1}{\theta_i} + \lambda \pi_i = 0 \text{ for } i = I$$

$2I + I - 1 + 1$ unknowns: $\{c_i, y_i\}_{i=1}^{I}$, $\{\mu_i\}_{i=2}^{I}$, λ

$2I$ FOCs + I constraints: ($I - 1$ IC constraints and the resource constraint)

This problem will have a solution. How can we decentralize it? We need to come up with a tax system such that taxes depend on earnings, and all agents are on their FOC

$$u'(c_i)\theta_i(1 - T'(y_i)) = v'\left(\frac{y_i}{\theta_i}\right)$$

and

$$c_i = y_i - T(y_i)$$

Note that marginal and average tax rates are only exactly pinned down at grid points.

FOCs

$$c_i \quad : \quad W_i \pi_i u'(c_i) + \pi_i \mu_i u'(c_i) - \pi_{i+1} \mu_{i+1} u'(c_i) - \lambda \pi_i = 0$$

$$y_i \quad : \quad -W_i \pi_i v'\left(\frac{y_i}{\theta_i}\right) \frac{1}{\theta_i} - \pi_i \mu_i v'\left(\frac{y_i}{\theta_i}\right) \frac{1}{\theta_i} + \pi_{i+1} \mu_{i+1} v'\left(\frac{y_i}{\theta_{i+1}}\right) \frac{1}{\theta_{i+1}} + \lambda \pi_i = 0$$

$$u'(c_i)\theta_i \frac{W_i \pi_i + \pi_i \mu_i - \pi_{i+1} \mu_{i+1}}{W_i \pi_i + \pi_i \mu_i - \pi_{i+1} \mu_{i+1} \frac{v'\left(\frac{y_i}{\theta_{i+1}}\right) \frac{1}{\theta_{i+1}}}{v'\left(\frac{y_i}{\theta_i}\right) \frac{1}{\theta_i}}} = v'\left(\frac{y_i}{\theta_i}\right)$$

3 Zero Marginal Tax at the Top, Positive Elsewhere

Note that for $i = I$ the $i+1$ terms are absent, so the FOCs collapse to:

$$(W_I \pi_I + \pi_I \mu_I)\, \theta_I u'(c_I) = (W_I \pi_I + \pi_I \mu_I)\, v' \left(\frac{y_I}{\theta_I} \right)$$

Compare this to the FOC for the decentralized economy in which individuals face a tax on earnings. It is clear that in this decentralization it must be the case that $T'(y_I) = 0$, so there is no distortion / wedge / implicit tax at the top. This is a classic result in the literature

More generically, suppose the v function is given by $v(x) = (1+\sigma)^{-1} x^{1+\sigma}$, so $v'(x) = x^{\sigma}$

$$v' \left(\frac{y_i}{\theta_{i+1}} \right) = v' \left(\frac{y_i}{\theta_i} \frac{\theta_i}{\theta_{i+1}} \right) = v' \left(\frac{y_i}{\theta_i} \right) \left(\frac{\theta_i}{\theta_{i+1}} \right)^{\sigma}$$

Then the FOC is

$$u'(c_i)\theta_i \underbrace{\left(\frac{W_i \pi_i + \pi_i \mu_i - \pi_{i+1}\mu_{i+1}}{W_i \pi_i + \pi_i \mu_i - \pi_{i+1}\mu_{i+1} \left(\frac{\theta_i}{\theta_{i+1}} \right)^{1+\sigma}} \right)}_{wedge = 1 - tax'} = v' \left(\frac{y_i}{\theta_i} \right)$$

It is clear from this that the term labeled 'wedge' is less than one (if $\mu_{i+1} > 0$), so the implicit tax rate is positive. It follows immediately from this that under the optimal Mirrleesian scheme, labor supply is distorted, and the allocation is not equal to the first best.

What is the intuition for this positive marginal tax rate result? The planner wants the more productive type to produce more output (that is efficient), but not to consume more (given a utilitarian objective). If consumption is rising more slowly than income between income levels y_i and y_{i+1}, the marginal tax rate must be positive, on average, in this range. Suppose the marginal rate was zero for type $i+1$ right at income level y_{i+1}. Type $i+1$ would then not be tempted to work slightly less: on the margin, their consumption would fall one for one with income. But this is just another way of saying that $i+1's$ incentive constraint would be slack at a zero marginal rate, and the planner could therefore increase downward redistribution without violating incentive constraints.

4 Numerical Solution

How do we solve this problem numerically? We could simply feed a non-linear solver the entire system of equations. Here is a possibly more efficient sketch of an approach.

1. Guess λ

2. Guess c_1
3. Solve for μ_2 from FOC for c_1
4. Solve for y_1 from FOC for y_1
5. Solve a system of 3 equations (2 FOCs and IC_2) in 3 unknowns to solve for c_2, y_2 and μ_3
6. Iterate upwards through the grid
7. At $I-1$ we solve for c_{I-1}, y_{I-1}, μ_I
8. Then use 2 FOCs at I to solve for c_I and y_I
9. Check IC_I and adjust c_1 if not satisfied
10. Check resource constraint and adjust λ if not satisfied.

5 Diamond Saez Equation

We have

$$\begin{aligned}
(1-\tau_i) &= \frac{W_i\pi_i + \pi_i\mu_i - \pi_{i+1}\mu_{i+1}}{W_i\pi_i + \pi_i\mu_i - \pi_{i+1}\mu_{i+1}\left(\frac{\theta_i}{\theta_{i+1}}\right)^{1+\sigma}} \\
\tau_i &= \frac{\pi_{i+1}\mu_{i+1} - \pi_{i+1}\mu_{i+1}\left(\frac{\theta_i}{\theta_{i+1}}\right)^{1+\sigma}}{W_i\pi_i + \pi_i\mu_i - \pi_{i+1}\mu_{i+1}\left(\frac{\theta_i}{\theta_{i+1}}\right)^{1+\sigma}} \\
&= \frac{\mu_{i+1}\pi_{i+1}\left(1-\left(\frac{\theta_i}{\theta_{i+1}}\right)^{1+\sigma}\right)v'\left(\frac{y_i}{\theta_i}\right)}{\lambda\pi_i\theta_i}
\end{aligned}$$

where the last line subsitutes in the FOC for y_i.

Now let's solve for the multipliers in terms of allocations. The FOC for c_i is

$$W_i\pi_i u'(c_i) + \pi_i\mu_i u'(c_i) - \pi_{i+1}\mu_{i+1}u'(c_i) = \lambda\pi_i$$

Dividing by $u'(c_i)$ and summing across the FOCs for all i gives

$$\begin{aligned}
\sum_i W_i\pi_i &= \lambda\sum_i \frac{\pi_i}{u'(c_i)} \\
\lambda &= \frac{1}{\sum_i \frac{\pi_i}{u'(c_i)}}
\end{aligned}$$

From the same FOC

$$
\begin{aligned}
\left(W_i\pi_i + \pi_i\mu_i - \pi_{i+1}\mu_{i+1}\right) u'(c_i) &= \lambda\pi_i \\
\pi_i\mu_i &= \frac{\lambda\pi_i}{u'(c_i)} - W_i\pi_i + \pi_{i+1}\mu_{i+1} \\
\pi_i\mu_i &= \frac{\lambda\pi_i}{u'(c_i)} - W_i\pi_i + \left(\frac{\lambda\pi_{i+1}}{u'(c_{i+1})} - W_{i+1}\pi_{i+1} + \pi_{i+2}\mu_{i+2}\right) \\
&= \lambda\sum_i^I \frac{\pi_i}{u'(c_i)} - \sum_i^I W_i\pi_i
\end{aligned}
$$

Plugging the expression for $\pi_i\mu_i$ into the expression for τ_i we have

$$
\begin{aligned}
\tau_i &= \frac{\mu_{i+1}\pi_{i+1}\left(1-\left(\frac{\theta_i}{\theta_{i+1}}\right)^{1+\sigma}\right)v'\left(\frac{y_i}{\theta_i}\right)}{\lambda\pi_i\theta_i} \\
&= \frac{\left(\lambda\sum_{s=i+1}^I \frac{\pi_s}{u'(c_s)} - \sum_{s=i+1}^I W_s\pi_s\right)\left(1-\left(\frac{\theta_i}{\theta_{i+1}}\right)^{1+\sigma}\right)v'\left(\frac{y_i}{\theta_i}\right)}{\lambda\pi_i\theta_i}
\end{aligned}
$$

We also know that

$$
\begin{aligned}
u'(c_i)\theta_i(1-\tau_i) &= v'\left(\frac{y_i}{\theta_i}\right) \\
(1-\tau_i) &= \frac{v'\left(\frac{y_i}{\theta_i}\right)}{u'(c_i)\theta_i}
\end{aligned}
$$

so

$$
\begin{aligned}
\frac{\tau_i}{1-\tau_i} &= \frac{u'(c_i)}{\pi_i}\left(\sum_{s=i+1}^I \frac{\pi_s}{u'(c_s)} - \frac{1}{\lambda}\sum_{s=i+1}^I W_s\pi_s\right)\left(1-\left(\frac{\theta_i}{\theta_{i+1}}\right)^{1+\sigma}\right) \\
&= \frac{u'(c_i)}{\pi_i}\left(\sum_{s=i+1}^I \frac{\pi_s}{u'(c_s)} - \sum_i \frac{\pi_i}{u'(c_i)}\sum_{s=i+1}^I W_s\pi_s\right)\left(1-\left(\frac{\theta_i}{\theta_{i+1}}\right)^{1+\sigma}\right) \\
&= \left(1-\left(\frac{\theta_i}{\theta_{i+1}}\right)^{1+\sigma}\right)\frac{1}{\pi_i}\left(\sum_{s=i+1}^I \pi_s\left(\frac{u'(c_i)}{u'(c_s)} - \sum_i \frac{\pi_i}{u'(c_i)}u'(c_i)W_s\right)\right)
\end{aligned}
$$

Now suppose the underlying productivity distribution is truly continuous, with CDF $F(\theta)$ and PDF $f(\theta)$.

Suppose that to construct our discrete approximation we have defined the mass points on the grid as $\pi_i = f(\theta_i)(\theta_{i+1} - \theta_i)$

Let's now construct the continuous productivity version of our discrete Diamond-Saez equation.

The second term is straightforward

$$\sum_{s=i+1}^{I} \pi_s \left(\frac{u'(c_i)}{u'(c_s)} - \sum_i \frac{\pi_i}{u'(c_i)} u'(c_i) W_s \right) \rightarrow \int_{s=\theta}^{\infty} \left(\frac{u'(c(\theta))}{u'(c(s))} - \int \frac{1}{u'(c(\theta))} dF(\theta) \cdot u'(c(\theta)) W(s) \right) dF_\theta(s)$$

What happens to the first term, $\left(1 - \left(\frac{\theta_i}{\theta_{i+1}}\right)^{1+\sigma}\right) \frac{1}{f(\theta_i)(\theta_{i+1}-\theta_i)}$ as the grid approaches the continuous limit, i.e., $\frac{\theta_{i+1}}{\theta_i} \rightarrow 1$?

By l'Hoptital's rule, it converges to $(1+\sigma)\frac{1}{\theta_i f(\theta_i)}$

Thus, we have an implicit solution for optimal marginal tax rates at each productivity value θ :

$$\frac{\tau(\theta)}{1-\tau(\theta)} = (1+\sigma) \frac{1}{\theta f(\theta)} \int_{s=\theta}^{\infty} \frac{u'(c(\theta))}{u'(c(s))} \left(1 - \int \frac{1}{u'(c(\theta))} dF(\theta) \cdot u'(c(s)) W(s) \right) dF_\theta(s)$$

Look at the second term in parentheses. This captures a distributional incentive to set marginal rates high.

In particular, imagine raising the marginal rate a little at some productivity value θ

$u'(c(s))W(s)$ will be declining in s.

So the entire integral is maximized at $s = \theta$ s.t. $W(\theta)u'(c(\theta)) = \int \frac{1}{u'(c(\theta))} dF(\theta)$

The equation indicates additional considerations

The larger is σ, the higher will be marginal rates, all else equal (higher σ => labor supply less elastic => taxes less distortionary)

The larger is $\theta f(\theta)$ the lower marginal tax rates will be at θ (high density => lots of agents choices distorted by higher marginal rates).

But there is a limit to how much we can learn from staring at this equation, because the consumption schedule is endogenous and depends on the tax schedule. So we have taxes on both sides of the equation.

Suppose we assume $u(c) = \log c$. Then the Diamond Saez equation becomes

$$\begin{aligned} \frac{\tau(\theta)}{1-\tau(\theta)} &= (1+\sigma) \frac{1}{\theta f(\theta)} \int_{s=\theta}^{\infty} \frac{c(s)}{c(\theta)} \left(1 - \frac{W(s)}{c(s)} C \right) dF_\theta(s) \\ &= (1+\sigma) \frac{1}{\theta f(\theta)} \int_{s=\theta}^{\infty} \frac{1}{c(\theta)} \left(c(s) - W(s) C \right) dF_\theta(s) \end{aligned}$$

What if we assume $u(c) = c$? (very non-standard in macro, very common in public finance)

Now our tax expression simplifies to

$$\frac{\tau(\theta)}{1-\tau(\theta)} = (1+\sigma)\frac{1}{\theta f(\theta)} \int_{s=\theta}^{\infty} (1 - W(s))\, dF_\theta(s)$$

If $W(s) = 1$ for all s, then zero marginal tax rates are optimal.

If $W(s)$ is decreasing in s, then want positive tax rates.

6 Homework

Economy with three types

Productivity	0.5	1.0	2.5
Population share	0.3	0.6	0.1

Utility function is

$$\log c - \frac{h^{1+\sigma}}{1+\sigma}$$

$\sigma = 2$

No government purchases

Solve for optimal allocations for (i) utilitarian planner, (ii) planner that only cares about one of the 3 types, for each of the three types