# Notes on Solving the Kocherlakota (1996) Economy

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Suppose there are equal numbers of two types of households

Households have preferences over consumption given by

$$E\sum_{t=0}^{\infty}\beta^t u(c_t)$$

where $u$ is strictly concave, $\beta < 1$.

Each hosuehold receives a stochastic income stream, $\{y_t\}_{t=0}^{\infty}$, where $y_t$ is *iid* according to the discrete probability distribution $Prob(y_t = y_s \in [0,1]) = \Pi_s$, $s \in \{1, 2, ..., S\}$.

(later we will loosen the *iid* assumption)

Assume that if one type of household receives $y_s$, the other receives $1 - y_s$.

At any date $t$, type 1 household has income $y_t$ and consumption $c_t$ while type 2 household has $1 - y_t$ and $1 - c_t$.

Think about set of efficient incentive-compatible allocations, when households always have the option to revert to autarky.

## 1 Approach 1 (see Ljunqvist and Sargent)

Look for a recursive formulation by finding a state variable $v_t$ such that efficient allocations have the form

$$c_t = g(v_t, y_t)$$

where $v_t$ evolves according to

$$v_{t+1} = l(v_t, y_t)$$

$v_t$ must summarize, in an appropriate fashion, the history $y^t = \{y_t\}_{t=0}^{t}$.

Try letting $v_t$ be expected discounted lifetime utility from period $t$ onwards promised to type 1 agent in $t - 1$.

Notation:

- $v$ : expected discounted lifetime utility promised to type 1 agent.

- $P(x)$ : maximum expected discounted lifetime utility that can be promised to type 2 agent given $v = x$
- $w_s$ : expected continuation utility from tomorrow on for the type 1 agent if the current shock turns out to be $s$.
- $c_s$ : consumption for the type 1 agent if the current shock turns out to be $s$.

Suppose the planner makes choices for $c_s$ (and implicitly $1-c_s$) and $w_s$ prior to the realization of $s$ in the current period.

Planner's problem in recursive form

$$P(v) = \max_{\{c_s, w_s\}} \sum_{s=1}^{S} \Pi_s \{u(1-c_s) + \beta P(w_s)\}$$

subject to:

1. The type 1 guy gets at least $v$ in terms of expected utility

$$\sum_{s=1}^{S} \Pi_s [u(c_s) + \beta w_s] \geq v$$

2. The type 1 guy wants to participate

$$u(c_s) + \beta w_s \geq u(y_s) + \beta v_{aut} \qquad s = 1, ..., S$$

3. The type 2 guy wants to participate

$$u(1-c_s) + \beta P(w_s) \geq u(1-y_s) + \beta v_{aut} \qquad s = 1, ..., S$$

$$c_s \in [0,1]$$

$$w_s \in [v_{aut}, v_{\max}]$$

Here $v_{aut}$ is the value of autarky, and because of symmetry is the same for both types

$v_{\max}$ is the maximum continuation utility a type 1 agent can be promised subject to the type 2 agent being willing to participate.

Associate multipliers $\mu$, $\{\lambda_s\}_{s=1}^S, \{\theta_s\}_{s=1}^S$ with constraints 1, 2 and 3. Assuming $P(v)$ is concave and differentiable, the first order conditions are

$$\begin{aligned} c_s &: \quad -(\Pi_s + \theta_s)u'(1-c_s) + (\Pi_s\mu + \lambda_s)u'(c_s) = 0 \\ \frac{u'(1-c_s)}{u'(c_s)} &= \frac{\Pi_s\mu + \lambda_s}{\Pi_s + \theta_s} \end{aligned}$$

$$\begin{aligned} w_s &: \quad \beta\Pi_s\left(P'(w_s) + \mu\right) + \beta\lambda_s + \beta\theta_s P'(w_s) = 0 \\ \Pi_s P'(w_s) + \theta_s P'(w_s) &= -\left(\Pi_s\mu + \lambda_s\right) \\ P'(w_s) &= \frac{-\left(\Pi_s\mu + \lambda_s\right)}{\Pi_s + \theta_s} \end{aligned}$$

Combining the two FOCs

$$-\frac{u'(1-c_s)}{u'(c_s)} = P'(w_s)$$

This says that agents' MRS's between $c_s$ and $w_s$ are equalized (equivalently the two agents are equally willing to trade off $c_s$ and $w_s$)

$$\frac{u'(c_s)}{\beta} = -\frac{u'(1-c_s)}{\beta P'(w_s)}$$

By the envelope theorem

$$P'(v) = -\mu$$

For a given $v$ and a given $s$, at most one participation cosntraint can bind. Thus there are 3 possilibities:

1. Neither constraint binds:. $\lambda_s = \theta_s = 0$ implies

$$u'(1-c_s) = \mu u'(c_s)$$

$$P'(w_s) + \mu = 0$$

Thus

$$\frac{u'(1-c_s)}{u'(c_s)} = -P'(w_s) = \mu$$

So in this case, consumption is independent of the endowment, and since $P'(v) = -\mu = P'(w_s)$, $w_s = v$.

2. $\lambda_s > 0$ and $\theta_s = 0$. Now the second FOC is

$$\Pi_s\left(P'(w_s) + \mu\right) + \lambda_s = 0$$

which implies

$$\Pi_s\left(P'(w_s) - P'(v)\right) + \lambda_s = 0$$

Since $P'(w_s) < P'(v)$, $w_s > v$ and thus both promised value and consumption of the type 1 agent, $c_s$, are increased, since

$$\frac{u'(1-c_s)}{u'(c_s)} = -P'(w_s)$$

The interpretation is that the type 1 agent gets more consumption today and is promised more value in the future in exchange for giving some of his endowment to the planner (and thereby to agent 2)

3. $\lambda_s = 0$ and $\theta_s > 0$. Now

$$\Pi_s \left(P'(w_s) + \mu\right) + \theta_s P'(w_s) = 0$$

$$\Pi_s P'(v) = \theta_s P'(w_s) + \Pi_s P'(w_s)$$

$$P'(w_s) = \frac{\Pi_s P'(v)}{\theta_s + \Pi_s}$$

In this case we have the reverse situation. The planner must lower both $w_s$ and $c_s$.

Given $v$, the first case prevails for intermediate values for $y_t$, the second in high- $y_t$ states and the third in low $y_t$ states.

The optimal contract expresses $c_s$, $w_s$ as non-decreasing functions of $y_s$ with the property that there are 2 numbers $y_l(v)$ and $y_h(v)$ (both increasing in $v$) such that $c_s$, $w_s$ are each constant for $y_s \in [y_l(v), y_h(v)]$, i.e. risk sharing is complete for shocks in this range.

# 2 Approach 2: Marcet-Marimon Lagrangian approach (see also Kehoe and Perri, ECA 2002)

I now describe an alternative recursive approach to solving for efficient allocations.

Suppose now that shocks are potentially persistent and follow a first-order Markov process.

Let $y[y^t]$ denote the last element of $y^t$.

Suppose the planner puts weight $1-\lambda$ on the agent who receives endowment $1 - y[y^t]$ and weight $\lambda$ on the agent who receives $y[y^t]$.

Let $V_{aut}(y[y^t])$ denote the value of autarky for the type $\lambda$ agent following history $h^t$, and $V_{aut}^*(y[y^t])$ denote the value for the type $1-\lambda$ agent (note that the value of autarky depends only on $y[y^t]$ given the Markov assumption).

The Lagrangian for the planner is

$$\begin{aligned} J &= (1-\lambda)\sum_{t=0}^{\infty}\sum_{y^t}\beta^t\pi(y^t)u(1-c(y^t)) + \lambda\sum_{t=0}^{\infty}\sum_{y^t}\beta^t\pi(y^t)u(c(y^t)) + \\ & \sum_{t=0}^{\infty}\sum_{y^t}\beta^t\pi(y^t)\mu(y^t)\left[\sum_{j=0}^{\infty}\sum_{y^{t+j}}\beta^j\pi(y^{t+j}|y[y^t])u(c(y^{t+j})) - V_{aut}(y[y^t])\right] + \\ & \sum_{t=0}^{\infty}\sum_{y^t}\beta^t\pi(y^t)\eta(y^t)\left[\sum_{j=0}^{\infty}\sum_{y^{t+j}}\beta^j\pi(y^{t+j}|y[y^t])u(1-c(y^{t+j})) - V^*_{aut}(y[y^t])\right] \end{aligned}$$

where, for example $\beta^t\pi(y^t)\mu(y^t)$ is the multiplier on the date $t$ history $y^t$ incentive compatibility constraint for the type $\lambda$ agent.

Now there is a useful trick called the 'partial summation' formula which says that

$$\sum_{t=0}^{\infty}\beta^t\mu_t\sum_{j=0}^{\infty}\beta^j u(c_{t+j}) = \sum_{t=0}^{\infty}\beta^t M_t u(c_t)$$

where $M_t = M_{t-1} + \mu_t$ with $M_{-1} = 0$

Verifying this

$$\begin{aligned} \sum_{t=0}^{\infty}\beta^t\mu_t\sum_{j=0}^{\infty}\beta^j u(c_{t+j}) &= \mu_0\left[\sum_{j=0}^{\infty}\beta^j u(c_j)\right] + \beta\mu_1\left[\sum_{j=0}^{\infty}\beta^j u(c_{1+j})\right] + \beta^2\mu_2\left[\sum_{j=0}^{\infty}\beta^j u(c_{2+j})\right] \\ &= \mu_0 u(c_0) + (\mu_0+\mu_1)\beta u(c_1) + (\mu_0+\mu_1+\mu_2)\beta^2 u(c_2) + ... \\ &= M_0 u(c_0) + M_1\beta u(c_1) + M_2\beta^2 u(c_2) + ... \end{aligned}$$

We can use this trick to rewrite the Lagrangian:

$$\begin{aligned} J &= (1-\lambda)\sum_{t=0}^{\infty}\sum_{y^t}\beta^t\pi(y^t)u(1-c(y^t)) + \lambda\sum_{t=0}^{\infty}\sum_{y^t}\beta^t\pi(y^t)u(c(y^t)) + \\ & \sum_{t=0}^{\infty}\sum_{y^t}\beta^t\pi(y^t)\left[M(y^t)u(c(y^t)) - \left[M(y^t)-M(y^{t-1})\right]V_{aut}(y[y^t])\right] + \\ & \sum_{t=0}^{\infty}\sum_{y^t}\beta^t\pi(y^t)\left[N(y^t)u(1-c(y^t)) - \left[N(y^t)-N(y^{t-1})\right]V^*_{aut}(y[y^t])\right] \end{aligned}$$

where $M(y^t) = M(y^{t-1}) + \mu(y^t)$ with $M(y^{-1}) = 0$ and $N(y^t) = N(y^{t-1}) + \eta(y^t)$ with $N(y^{-1}) = 0$.

Now we can simplify slightly, if we let $M(y^{-1}) = \lambda$ and $N(y^{-1}) = 1-\lambda$, so then we get

$$\begin{aligned} J &= \sum_{t=0}^{\infty}\sum_{y^t}\beta^t\pi(y^t)\left[M(y^t)u(c(y^t)) - \left[M(y^t)-M(y^{t-1})\right]V_{aut}(y[y^t])\right] + \\ & \sum_{t=0}^{\infty}\sum_{y^t}\beta^t\pi(y^t)\left[N(y^t)u(1-c(y^t)) - \left[N(y^t)-N(y^{t-1})\right]V^*_{aut}(y[y^t])\right] \end{aligned}$$

Taking a first order condition with respect to $c(y^t)$ gives

$$\beta^t \pi(y^t) M(y^t) u'(c(y^t)) - \beta^t \pi(y^t) N(y^t) u'(1 - c(y^t)) = 0$$

Rearranging gives

$$\frac{N(y^t)}{M(y^t)} = \frac{u'(c(y^t))}{u'(1 - c(y^t))}$$

Now let us guess that optimal allocations in this economy can be described recursively, given as state variables at date $t$, $y[y^t]$ and $\frac{N(y^{t-1})}{M(y^{t-1})} = z(y^{t-1})$.

So let the state variable be $x = (z_{-1}, y)$ . Thus we will look for an allocation rule of the form $c = c(x)$, and a law of motion for $z$ of the form $z = z(x)$. Let the notation $z_{-1}[x]$ indicate the first element of $x$ and $y[x]$ indicate the second element.

Define $v(x) = \frac{\mu(x)}{M(x)}$ and $w(x) = \frac{\eta(x)}{N(x)}$.

We can show that

$$z(x) = \frac{1 - v(x)}{1 - w(x)} z_{-1}[x]$$

Check:

$$\frac{1 - v(x)}{1 - w(x)} z_{-1}[x] = \frac{\frac{M(x) - \mu(x)}{M(x)}}{\frac{N(x) - \eta(x)}{N(x)}} z_{-1}[x] = \frac{\frac{M(x_{-1})}{M(x)}}{\frac{N(x_{-1})}{N(x)}} \frac{N(x_{-1})}{M(x_{-1})}$$

So, given the functions $v(x)$ and $w(x)$ we have a law of motion for $z$.

We also have, implicitly, a decision rule $c(x)$ defined by

$$z(x) = \frac{u'(c(x))}{u'(1 - c(x))}$$

# 3 Case 1: No enforcement problems

Suppose, to get warmed up, that the government faces no enforcement problems. Then

$$v(x) = w(x) = 0 \quad \forall x$$

So

$$z(x) = z_{-1}[x] \quad \forall x$$

and

$$z_{-1}[x] = \frac{u'(c(x))}{u'(1 - c(x))} \quad \forall x$$

So consumption is constant through time.

To simulate the economy through time, the only additional thing we need to know is the initial value for $z$, or, equivalently, for $c$. We can solve for this by noticing that the planner will want to set

$$(1 - \lambda) u'(1 - \overline{c}) = \lambda u'(\overline{c})$$

where $\overline{c}$ is the value for the type $\lambda$ agent's consumption that will prevail for ever.

## 4 Case 2: Enforcement problems

Now consider the version with the enforcement problem

To make more progress, define value functions

$$W(x) = u(c(x)) + \beta \sum_{y'} \pi(y'|y) W(x')$$

Let $W^*(x)$ denote the foreign agent's value function.

Our solution strategy will be to make initial guesses for the functions $W(x)$, $W^*(x)$, $z(x)$ and $c(x)$ and to iterate on these guesses.

In fact we will create a (two-dimensional) grid over $x$, guess the values of these functions at grid points, and make some assumptions about what the functions look like in between grid points (e.g. that they are piecewise linear).

Let us use as initial guesses the solution to the problem without enforcement problems. In fact it is important that our initial guesses for the value functions are everywhere at least as large as the true solutions. We will return to this point.

So $z_0(x) = z_{-1}[x]\ \forall x$, etc.

Now take the first point in the grid on $x$, denoted $x_1$, and check whether enforcement constraints are satisfied, one constraint at a time.

So first check whether

$$u(c_0(x_1)) + \beta \sum_{y'} \pi(y'|y[x_1]) W_0(z_{-1}[x_1], y') > V_{aut}(y[x_1])$$

Then check the other agent's constraint.

To perform this check we first need to compute $V_{aut}(y[x_1])$ and $V^*_{aut}(y[x_1])$, which is straightforward.

There are three possible cases:

1. Neither agent's constraint is violated at $x_1$

   (a) Update functions as follows:

$$z_1(x_1) = z_{-1}[x_1]$$

   $c_1(x_1)$ is given by the solution to

$$z_1(x_1) = \frac{u'(c_1(x_1))}{u'(1 - c_1(x_1.))}$$

   and

$$\begin{aligned} W_1(x_1) &= u(c_1(x_1)) + \beta \sum_{y'} \pi(y'|y[x_1]) W_0(z_1(x_1), y') \\ W_1^*(x_1) &= u(1 - c_1(x_1)) + \beta \sum_{y'} \pi(y'|y[x_1]) W_0^*(z_1(x_1), y') \end{aligned}$$

2. The $\lambda-$type agent's constraint is violated at $x_1$ (and the $1-\lambda$ type agent's constraint is not violated).

   (a) We know that

   $$u(c_1(x_1)) + \beta \sum_{y'} \pi(y'|y[x_1]) W_0(z_1(x_1), y') = V_{aut}(y[x_1])$$

   $$z_1(x_1) = \frac{u'(c_1(x_1))}{u'(1 - c_1(x_1))}$$

   These are two equations in two unknowns: $c_1(x_1)$ and $z_1(x_1)$. Solve for these unknowns.

   (b) Given $c_1(x_1)$ and $z_1(x_1)$ we can update the value functions to get $W_1(x_1)$ and $W_1^*(x_1)$

3. The $1-\lambda$ type agent's constraint is violated at $x_1$ (and the $\lambda$ type agent's constraint is not violated). We deal with this case following steps very similar to $a$ and $b$ above.

Once we have finished with the first point in the grid on $x$, we move to $x_2$. We should still use the original functions $W_0$, $W_0^*$ and $z_0$ when solving for $c_1(x_2)$, $z_1(x_2)$, $W_1(x_2)$ and $W_1^*(x_2)$.

Once we have gone through the whole grid over $x$, we check whether $c_1(x) = c_0(x)\ \forall x$. If so, we are done. If not set $W(x) = W_1(x)$, $W^*(x) = W_1^*(x)$, $c(x) = c_1(x)$, $z(x) = z_1(x)$, and repeat.