Homework 3, due in class on November 23rd

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Consider the two agent planner's problem studied by Kocherlakota.

Suppose the two agents receive stochastic endowment streams, such that the aggregate endowment is 1 in every period. Endowments cannot be stored or disposed of. Each period, agent 1's endowment is drawn from a set $S = \{0.5 - \varepsilon, 0.5, 0.5 + \varepsilon\}$. Denote the elements of this set s_1, s_2 and s_3. Let s_t denote the draw from S at date t. Let $s^t = (s_0, s_1, s_2, ...s_t) \in S^t$ denote the history of the economy up to and including date t. Denote agent 1's endowment $y(s^t)$. Thus agent 2's endowment is $1 - y(s^t)$. Let the date zero probability of history s^t be denoted $\pi(s^t)$. Assume that $\pi(s^t|s^{t-1}) = \pi(s_t|s_{t-1})$ (i.e. productivity shocks are first order Markov). Assume that transition probabilities are given by the matrix:

$$\Gamma = \begin{pmatrix} q & 1-q & 0 \\ \frac{1-p}{2} & p & \frac{1-p}{2} \\ 0 & 1-q & q \end{pmatrix}$$

where the element $\Gamma_{i,j}$ is equal to $\pi(s_j|s_i)$, and $p \in (0,1)$, $q \in (0,1)$.

Assume that preferences for agent 1 are given by $\sum_t \beta^t \sum_{s^t} \pi(s^t) \log(c(s^t))$. Preferences for agent 2 are similar. Note that in equilibrium, the consumption of the type two agent is given by $1 - c(s^t)$. Assume that the planner cares about both types equally.

1. Assume that half of the mass of the ergodic distribution across S implied by the matrix Γ is on s_2. Given this assumption, derive an expression for q as a function of p. Thus there are three independent parameters in the model: : β, ε, and p

2. Assume, to start, that agents are not allowed to reject allocations proposed by the planner. Describe the planner's problem, and characterize the solution to it.

3. Now suppose that at any date, agents have the option of rejecting the allocations proposed by the planner and reverting to permanent autarky. Once in autarky, allocations are given by $c(s^t) = y(s^t)\ \forall t, s^t$. Define $V_i(s^t)$ to be the value for agent i of reverting to autarky at t given s^t.

   (a) Suppose $\beta = 0.96$, $p = 0.9$, and $\varepsilon = 0.25$. For these parameter values, compute the set of possible values for $V_i(s^t)$.

(b) Formulate (as a Lagrangian) the planner's problem now that the planner has to worry about the commitment problem. Use the Marcet and Marimon trick to formulate the problem in terms of the sum of the past values of the multipliers on the participation constraints. Take first order conditions. Now you will solve the model recursively, as we outlined in class. The state variable will be $x_t = \left(z(s^{t-1}), s_t\right)$ where $z(s^{t-1})$ is the ratio of the sum of the values of multipliers for the history s^{t-1}. Since neither t nor s^t is a state variable in the recursive formulation, we can simply write $x = (z_{-1}, s)$.

(c) Create a grid on z_{-1}. Let $Z = (z_1, z_2, ... z_n)$ denote the n points on this grid. To start, set $n = 11$. To space the grid points "evenly" so that $z_i < 1$, $i \leq 5$, $z_6 = 1$, $z_j > 1$, $j \geq 6$, use an exponential spacing formula:

$$z_i = z_{i-1} \left(\frac{z_n}{z_1}\right)^{\frac{1}{n-1}} \qquad 1 < i < n$$

How should one choose the end points for this grid? (hint: think about the possible range for the ratio of marginal utilities in autarky)

(d) Now you need initial guesses for decision variables, for the law of motion of the state variables, and for the value functions $W_i(x)$. Use the problem without enforcement constraints to produce these initial guesses given the parameter values above: $W_i^0(x)$, $z^0(x)$, $c^0(x)$, $v_i^0(x)$ $\forall x \in S \times Z$. ($v_i^0(x)$ is the ratio of the multiplier on agent i's incentive compatibility constraint to the sum of multipliers on this constraint).

4. Recall that the goal is to solve for constrained efficient allocations when the planner has to respect incentive compatibility constraints. You will do this by iterating on $W_i(x)$, $z(x)$, $c(x)$, $v_i(x)$ as follows:

(a) Take x_1 the first point on the grid over $X = Z \times S$. Check whether agent 1 has an incentive to default at this grid point. If he does set solve numerically for $c^1(x_1)$ and $z^1(x_1)$ to satisfy (i) the first order condition for c, and (ii) the incentive compatibility constraint for agent 1 with equality, where current period utility is given by $c^1(x_1)$ and continuation utility for each possible s' is given by $W_1^0(z^1(x_1), s')$. Note that $z^1(x_1)$ does not necessarily lie on the grid on Z. Thus you will want to interpolate (linearly) to evaluate the function $W_1^0(., s')$ in between grid points.

(b) If agent 1 does not want to default, check incentive compatibility for agent 2. If agent 2 wants to default, solve for $c^1(x_1)$ and $z^1(x_1)$ following an analogous procedure to the one above.

(c) If neither agent wants to default, $z^1(x_1) \equiv z^1(z_{-1} = z_1, s = s_1) = z_1$, $v_1^1(x_1) = v_2^1(x_1) = 0$, and $c^1(x_1)$ satisfies the FOC for c.

(d) Compute and store the new guess for the value functions

$$W_1^1(x_1) = u(c^1(x_1)) + \beta \sum_{s'} \pi(s'|s_1) W_1^0(z^1(x_1), s')$$

(similarly for $W_2^1(x_1)$)

(e) Move to x_2, the next point in the grid on X. Solve for decision rules at x_2. Repeat for every point in the grid on X.

(f) Now check whether $W_1^1(x) = W_1^0(x)$ for every $x \in X$. If the two are arbitrarily close, consider the value functions to have converged, and assume the decision rules are optimal. Otherwise, update the guessed value functions across the grid from $W^0(x)$ to the stored values $W^1(x)$ and resolve for decision rules across the entire grid. Repeat until convergence is achieved.

5. Now we want to check whether our solution appears reasonable, and whether we have achieved tolerable numerical accuracy. Thus check the following

   (a) $z(z_{-1} = 1, s = s_2) = 1$

   (b) $z_1 \leq z(x) \leq z_n$ for all x.

   (c) $z(., s)$ (weakly) increasing for all s

   (d) $z(z_{-1}, .)$ decreasing for all z_{-1}

   (e) $z(z_i, s = s_2) = 1/z(z_{11-i}, s = s_2)$, $z(z_i, s = s_1) = 1/z(z_{11-i}, s = s_3)$.

6. The next thing to do is to see whether we can reduce the width of the grid on z. Imagine the initial value for z is $z_{-1} = 1$. Now imagine that the endowment shock s_3 is drawn repeatedly. What is the minimum value for z that will be observed in equilibrium? Set z_1 slightly less than this value, set $z_n = 1/z_1$ and resolve for decision rules.

7. Simulate this economy for 100 periods, setting $z_{-1} = 1$ and $s_0 = s_2$ and drawing future values for s according to the transition probability matrix Γ. At each date in the simulation, first draw the new vector for the endowments, s_t. Then, given x_t, use the converged functions $c(x_t)$ to get consumption, and the function $z(x_t)$ to update the first element of x_{t+1}.

   (a) Plot the series for agent $1's$ endowment and his consumption. Is the planner able to achieve zero, partial or complete risk-sharing?

   (b) Plot the series for agent $1's$ value inside the contract, and his value of default through time. Do the same thing for agent 2.