Short notes on a simple global solution method[1]

Consider the following recursive maximization problem:

$$V(a,e) = \max_{c,a'} \left\{ u(c) + \beta \sum_{e' \in \{e_1,e_2\}} \pi(e'|e) V(a',e') \right\}$$

subject to

$$c \leq e + (1+r)a - a'$$
$$a' \geq -\phi$$

The Euler equation (substituting in the budget constraint) is

$$u'(e + (1+r)a - a') \geq \beta \sum_{e' \in \{e_1,e_2\}} \pi(e'|e) V_a(a',e')$$

with equality if $a' > 0$.

The Envelope condition is

$$V_a(a,e) = u'(e + (1+r)a - a')(1+r)$$

We are looking for a decision rule $a^*(a,e)$ for savings (given values for r and ϕ).

1. Construct a grid on $a = \{a_1, a_2 ... a_n\}$. Set $a_1 = -\phi$.
2. Guess a vector of values $\hat{V}_a$ for all combinations for a and e on the grid: $\hat{V}_a = \left\{ \hat{V}_a(a_1,e_1), \hat{V}_a(a_2,e_1), ..., \hat{V}_a(a_n,e_1), \hat{V}_a(a_1,e_2), ..., \hat{V}_a(a_n,e_2) \right\}$ (e.g. $\hat{V}_a = 1$)
3. Take the first point on the grid (a_1, e_1).
4. Check whether

$$u'(e_1 + (1+r)a_1 + \phi) - \beta \sum_{e' \in \{e_1,e_2\}} \pi(e'|e_1) \hat{V}_a(a_1,e') > 0.$$

[1] Jonathan Heathcote, May 13 2000, last updated April 2006.

- If it is then $a^*(a_1, e_1) = -\phi$
- If it is not then $a^*(a_1, e_1) > -\phi$. Solve for $a^*(a_1, e_1)$ as follows:
  1. Note that by concavity $u'()$ is increasing in a' while $\hat{V}_a(\ )$ is decreasing in a'.
  2. Assume that $\hat{V}_a$ is piecewise linear across the grid in the asset holding dimension.
  3. Find the grid point i^* such that[2]

$$u'(e_1 + (1+r)a_1 - a_{i^*}) - \beta \sum_{e' \in \{e_1, e_2\}} \pi(e'|e_1)\hat{V}_a(a_{i^*}, e') < 0$$

     and

$$u'(e_1 + (1+r)a_1 - a_{i^*+1}) - \beta \sum_{e' \in \{e_1, e_2\}} \pi(e'|e_1)\hat{V}_a(a_{i^*+1}, e') > 0$$

  4. Use a non-linear equation solver to solve for $a^*(a_1, e_1)$ as the solution to

$$\begin{aligned} & u'(e_1 + (1+r)a_1 - a^*(a_1, e_1)) \\ = \ & \beta \sum_{e'} \pi(e'|e_1) \left\{ \hat{V}_a(a_{i^*}, e') + \left(\frac{a^*(a_1, e_1) - a_{i^*}}{a_{i^*+1} - a_{i^*}} \right) \left[\hat{V}_a(a_{i^*+1}, e') - \hat{V}_a(a_{i^*}, e') \right] \right\} \end{aligned}$$

     (why do we need a non-linear equation solver? because while the RHS is linear in $a^*(a_1, e_1)$, the LHS is non-linear)

5. Move on to the next point in the grid (a_2, e_1) and repeat this process.

6. Iterate to get decision rules for all points in the grid.

7. Use the envelope condition to update the guesses for $\hat{V}_a$, i.e. for each point (a_i, e_j) in the grid, set $\hat{V}_a(a_i, e_j) = u'(we_j + (1+r)a_i - a^*(a_i, e_j))(1+r)$

8. If the new $\hat{V}_a$ is the same as the old $\hat{V}_a$ we are (almost) done. Otherwise repeat the whole process given the new $\hat{V}_a$

9. The last thing to check is whether our grid is appropriate. In particular if initial assets are between a_1 and a_n, will assets remain always within this interval? Thus upon convergence we need to check that $a^*(a_n, e_2) \leq a_n$. If not, increase a_n and return to step 2.

[2] If the following inequality is negative for $a_{i+1} = a_n$ then set $i^* = n - 1$.