MIDTERM, GRADUATE MACRO, ECON 606

Jonathan Heathcote

March 2nd 2006

Consider the following economy. There are two sectors: an apple sector, and an orange sector. Both sectors use land F and labor n to produce. The amount of land in each sector, F_o and F_a is fixed: $F_o = F_a = F$. The production technology is Cobb-Douglas:

$$Y_i = z_i F_i^{\theta} n_i^{1-\theta} \qquad i = a, o$$

where $0 \leq \theta \leq 1$. z_i, which determines sector-specific productivity, is given by

$$\begin{aligned} z_o &= 70 + (T - 70) \\ z_a &= 70 - (T - 70) \end{aligned}$$

where $T \in \{60, 61, ..., 79, 80\}$ is the average temperature in Fahrenheit in the period, and evolves over time according to a first-order Markov process defined by the transition probability matrix Π.

Infintely-lived consumers have identical preferences over a composite consumption good C and leisure l. Suppose, to start with, that labor markets are segmented: half of the population can only work on apple farms, the other half can only work on orange farms. Let superscripts denote the identity of workers / consumers: thus, for example, c_a^o denotes consumption of apples by workers who work on orange farms. Preferences for workers of type $i \in \{a, o\}$ are given by

$$E \sum_{t=0}^{\infty} \beta^t u(C_t^i, l_t^i)$$

where

$$\begin{aligned} u(C^i, l^i) &= u(C^i) + v(l^i) \\ C^i &= \left(\left(c_a^i\right)^{\frac{\sigma-1}{\sigma}} + \left(c_o^i\right)^{\frac{\sigma-1}{\sigma}} \right)^{\frac{\sigma}{\sigma=1}} \end{aligned}$$

$\sigma \geq 0$ is the elasticity of substitution between apples and oranges.

Assume the standard inequality constraints must be respected: $c_a^i, c_o^i, l^i \geq 0$, $l^i \leq 1$.

PART 1: Consider a planner who cares equally about apple workers and orange workers.

1. Write down a recursive formulation of the planner's problem. Take care to choose appropriate state variable(s).

$$V(T) = \max_{n^o,n^a,c_a^a,c_a^o,c_o^a,c_o^o} \left\{ \begin{array}{c} \frac{1}{2}\left[u\left(\left((c_a^o)^{\frac{\sigma-1}{\sigma}} + (c_o^o)^{\frac{\sigma-1}{\sigma}}\right)^{\frac{\sigma}{\sigma=1}}\right) + v(1-n^o)\right] \\ +\frac{1}{2}\left[u\left(\left((c_a^o)^{\frac{\sigma-1}{\sigma}} + (c_o^o)^{\frac{\sigma-1}{\sigma}}\right)^{\frac{\sigma}{\sigma=1}}\right) + v(1-n^0)\right] + \beta E_{|T}\left[V(T')\right] \end{array} \right\}$$

subject to

$$\begin{aligned} c_a^a + c_a^o &= z_a(T)F^\theta (n^a)^{(1-\theta)} \\ c_o^a + c_o^o &= z_o(T)F^\theta (n^o)^{(1-\theta)} \end{aligned}$$

and the inequality constraints, where

$$\begin{aligned} z_o(T) &= 70 + (T-70) \\ z_a(T) &= 70 - (T-70) \end{aligned}$$

2. What are the planner's first order conditions?

   Assume none of the inequality constraints are binding. Then the FOCs simplify to

$$\begin{aligned} u_{c_a^o} &= u_{c_a^a} \\ u_{c_o^o} &= u_{c_o^a} \\ (1-\theta)\frac{1}{2}u_{c_a^o}z_a(T)F^\theta (n^a)^{-\theta} &= v'(1-n^a) \\ (1-\theta)\frac{1}{2}u_{c_o^o}z_o(T)F^\theta (n^o)^{-\theta} &= v'(1-n^o) \end{aligned}$$

3. Compare the optimal consumption bundle for apple farm workers to the optimal consumption bundle for orange farm workers.

   The two types of workers get the same consumption bundle - $c_a^o = c_a^a$, $c_o^o = c_o^a$ (this follows directly from the first order conditions above). They get different amounts of leisure.

4. Consider the limiting case as $\sigma \to \infty$. Can you solve for the optimal value for the ratio $\frac{n^o}{n^a}$ as a function of the state variable(s)? If you can, do so; if not, explain what extra information would be required.

   As $\sigma \to \infty$, $u_{c_a^o} - u_{c_o^o} \to 0$. Thus the second pair of first order conditions can be combined to give

$$\frac{v'(1-n^a)}{z_a(T)\,(n^a)^{-\theta}} = \frac{v'(1-n^o)}{z_o(T)\,(n^o)^{-\theta}}$$

Without more assumptions, I don't think any more progress can be made towards solving for the ratio n^o/n^a. With some assumptions on v we could go further. For example, if v is linear, then

$$\frac{n^o}{n^a} = \left(\frac{z_o(T)}{z_a(T)}\right)^{\frac{1}{\theta}}$$

PART 2: Now consider a competitive equilibrium. Suppose the assets traded are shares in farms, and that the number of (infinitely-divisible) shares in each sector is equal to 1. Suppose farms maximize the expected present value of dividends, and that they value dividends across dates and states according to their worker's state-specific marginal rate of substitution for the good they produce.

5. Define the (two types of) consumers' problems and the firm's problem recursively. Take care in choosing appropriate state variables for the consumers problems, and for the firm's problem.

   Let S_a^o denote the (aggregate) shares in apple-producing firms held by orange-producing workers, and adopt analogous notation for other share holding patterns. Let lower case letters denote individual shareholdings. Let $X = (T, S_o^o, S_a^a)$ denote the aggregate state variable. Let $p(X)$ be a function defining the price of apples relative to oranges, $q_o(X)$ the ex-dividend price of shares in orange-producers (in units of oranges), $q_a(X)$ the ex-dividend price of shares in apple-producers (in units of apples), $w_o(X)$ is the wage for orange workers (in oranges), $w_a(X)$ is the wage for apple workers (in apples), $d_o(X)$ is the dividend from orange-producers (in oranges) and $d_a(X)$ is the dividend from apple-producers (in apples).

   The problem for the representative orange worker is

$$V^o(s_o^o, s_a^o, X) = \max_{c_a^o, c_o^o, n^o, s_o^{o\prime}, s_a^{o\prime}} \left\{ u\left(\left((c_a^o)^{\frac{\sigma-1}{\sigma}} + (c_o^o)^{\frac{\sigma-1}{\sigma}}\right)^{\frac{\sigma}{\sigma=1}}\right) + v(1-n^o) + \beta E\left[V^o(s_o^{o\prime}, s_a^{o\prime}, X')\right] \right\}$$

   subject to

$$\begin{aligned} c_o^o + p(X)c_a^o + q_o(X)(s_o^{o\prime} - s_o^o) + p(X)q_a(X)(s_a^{o\prime} - s_a^o) &\leq w_o(X)n^o + d_o(X)s_o^o + p(X)d_a(X)s_a^o \\ 0 &\leq n^o \leq 1 \\ c_a^o, c_o^o &\geq 0 \end{aligned}$$

   taking as given Π and forecasting functions

$$\begin{aligned} S_o^{o\prime} &= G_o(X) \\ S_a^{a\prime} &= G_a(X) \end{aligned}$$

   The problem for the apple worker is similar.

   The representative orange-producing firm's problem is

$$W^o(X) = \max_{n_o} \left\{ z_o(T) F^\theta n_o^{1-\theta} - w_o(X)n_o + E\left[Q(X, X')W^o(X')\right] \right\}$$

where $Q(X, X')$ is the firm's discount factor. Note that the only choice variable for the firm is how much labor to rent, n_o, and this choice has no dynamic consequences. Thus the firm's problem simplifies to

$$\max_{n_o} \left\{ z_o(T) F^{\theta} n_o^{1-\theta} - w_o(X) n_o \right\}$$

The problem for the representative apple-producing firm is similar.

(Why are aggregate stock holdings part of the state space? Because the wealth of workers will determine their willingness to work, which will in turn affect the equilibrium prices and dividends).

6. Define a recursive competitive equilibrium for this economy.

A RCE is a set of value functions and decision rules for households $V^i(s_o^i, s_a^i, X)$, $c_o^i(s_o^i, s_a^i, X)$, $c_a^i(s_o^i, s_a^i, X)$, $n^i(s_o^i, s_a^i, X)$, $s_o^{i\prime}(s_o^i, s_a^i, X)$, $s_a^{i\prime}(s_o^i, s_a^i, X)$ for $i = o, a$, decision rules for representative firms $n_i(X)$, forecasting rules $G_i(X)$, wages $w_i(X)$, prices $p(X)$, $q_i(X)$, and dividends $d_i(X)$ such that: (i) value functions and associated decision rules solve household problems, (ii) $n_i(X)$ solve the firms' problems, and $d_i(X)$ are the associated dividends, (iii) markets clear (2 labor markets, 2 goods markets, 2 stock markets):

$$\begin{aligned} n_a(X) &= n^a(1 - S_o^o, S_a^a, T, S_o^o, S_a^a) \\ n_o(X) &= \ldots \end{aligned}$$

$$\begin{aligned} c_a^a(1 - S_o^o, S_a^a, T, S_o^o, S_a^a) + c_a^o(S_o^o, 1 - S_a^a, T, S_o^o, S_a^a) &= z_a(T) F^{\theta} n_a(T, S_o^o, S_a^a) \\ &\ldots \end{aligned}$$

$$\begin{aligned} s_a^{a\prime}(1 - S_o^o, S_a^a, T, S_o^o, S_a^a) + s_a^{o\prime}(S_o^o, 1 - S_a^a, T, S_o^o, S_a^a) &= 1 \\ &\ldots \end{aligned}$$

Suppose that at $t = 0$ consumers working on apple farms are endowed with half the shares in both sectors: $s_{o,0}^a = s_{a,0}^a = 0.5$. Suppose that $T_0 = 70$, and suppose the matrix Π is symmetric (so that, for example, beginning at $T = 70$, any given increase in temperature has the same probability as a fall in temperature of the same magnitude). Suppose that apples and oranges are perfect substitutes. Consider three alternative scenarios:

(a) The benchmark for which you just defined an equilibrium: stocks can be costlessly traded, but labor is not mobile between sectors (as in the planner's problem).

(b) Transactions costs associated with stock trade are so large, that there is never any trade in stocks (ie $s_{o,t}^a = s_{a,t}^a = 0.5$ for all $t \geq 0$). Labor is not mobile between sectors

(c) Transactions cost are large, so $s^a_{o,t} = s^a_{a,t} = 0.5$ for all t, but after observing shocks and offered wages, workers can decide where to work.

7. Under which of these three scenarios, if any, are allocations in the competitive equilibrium equal to the allocations that solve the planner's problem? Prove / explain your answers as best you can.

   In (a) and (c) allocations are as in the planner's problem. Here is the intuition. If apples and oranges are perfect substitutes, the planner just wants to equally divide total fruit produced each period. If fruit is split equally, so that

$$u_{c^o_a} = u_{c^a_a} = u_{c^o_o} = u_{c^a_o}$$

   then labor allocations will also be efficient (to verify this, take the consumer's intra-temporal FOCs, substitute in the firms FOCs to eliminate the wage, and use the equal-splitting condition above to reproduce the planner's efficiency condition for relative labor supply in the case $\sigma \to \infty$). Thus the key to decentralizing the planner's allocation is to replicate this equal split of total fruit period-by-period.

   In case (a) because labor is not mobile across sectors, labor earnings will not be equalized. However, given the Cobb-Douglas technology, dividends are simply equal to fraction θ of output, while labor earnings are fraction $(1-\theta)$. By picking a portfolio biased towards the sector they do not work in, agents can ensure that their income remains a constant fraction of world output.

   In case (b) allocations will not be the same as in the competitive equilibrium, since they cannot adjust portfolios to compensate for a lack of labor mobility.

   In case (c) workers will relocate across sectors until wages are equalized across sectors. Then labor earnings for the two types of workers will be the same, as will their asset income (given equal constant portfolios). Thus both types will receive equal shares of world output, as in the solution to the planner's problem.

8. Under scenario (a), characterize equilibrium portfolio choices.

   Equating total income for apple workers and orange workers (assuming constant portfolios) requires

$$(1-\theta)Y_o + S^o_o\theta Y_o + (1-S^a_a)\theta Y_a = (1-\theta)Y_a + (1-S^o_o)\theta Y_o + S^a_a\theta Y_a$$

   Assuming symmetry, $S^o_o = S^a_a = S$

$$(1-\theta)Y_o + S\theta Y_o + (1-S)\theta Y_a = (1-\theta)Y_a + (1-S)\theta Y_o + S\theta Y_a$$

   so

$$S = \frac{2\theta - 1}{2\theta} = 1 - \frac{1}{2\theta}$$

Thus if $\theta = 1$ (no labor income), equilibrium portfolios are $S_o^o = S_a^a = \frac{1}{2}$, but for $\theta < 1$ portfolios are biased in that $S_o^o = S_a^a < \frac{1}{2}$. Intuitively, because workers cannot switch sectors, they pool sector-specfic risk by biasing their stock portfolios towards the sector in which they cannot work.