Introduction to Linearization Methods

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1. Solving systems of stochastic linear difference equations (see Farmer's book)

1.1. Example of a first order stochastic difference equation.

$$x_{t+1} = b + ax_t + \varepsilon_{t+1}$$

ε is a random variable.

Conditions for stationary distribution of x

(i) $|a| < 1$

(ii) ε drawn from an invariant probability distribution

Solution to the model is a sequence of probability distributions $\{G_t(\cdot \mid x_0)\}_{t=1}^{\infty}$

1.2. Simple application: a stochastic version of the Solow growth model.

Equations describing the model:

$$Y_t = z_t F(K_t, N_t) \tag{1}$$

$$N_t = \gamma^t \overline{N}$$

$$K_{t+1} = (1-\delta)K_t + Y_t - C_t \tag{2}$$

$$C_t = (1-s)Y_t \tag{3}$$

$$z_{t+1} = (1-\rho) + \rho z_t + \varepsilon_{t+1} \qquad \varepsilon \tilde{} N(0, \sigma^2) \tag{4}$$

$$z_0, K_0, N_0 \text{ given}$$

1. Rewrite in terms of stationary variables. Divide 1, 2 and 3 by N_t, and for any variable x let $x_t = \frac{X_t}{N_t}$

$$y_t = z_t F(k_t, 1) = z_t f(k_t) \qquad \text{(assuming } F \text{ is homogenous of degree one)}$$

$$\gamma k_{t+1} = (1-\delta)k_t + y_t - c_t$$

$$c_t = (1-s)y_t$$

2. Compute the (non-stochastic) steady state

$$y^* = z^* f(k^*)$$

$$z^* = (1-\rho) + \rho z^*$$

$$\gamma k^* = (1-\delta)k^* + y^* - c^*$$

$$c^* = (1-s)y^*$$

3. Linearize around the steady state (take a first order Taylor series expansion)

$$y_t - y^* = z^* f'(k^*)\,(k_t - k^*) + f(k^*)(z_t - z^*)$$

$$z_{t+1} - z^* = \rho(z_t - z^*) + \varepsilon_{t+1} \qquad \varepsilon \tilde{} N(0, \sigma^2)$$

$$\gamma(k_{t+1} - k^*) = (1-\delta)\,(k_t - k^*) + (y_t - y^*) - (c_t - c^*)$$

$$c_t - c^* = (1-s)(y_t - y^*)$$

1.3. Review of some useful linear algebra. *Definition of eigenvalues and eigenvectors*

Consider an $n \times n$ matrix A that maps R^n into R^n.

The roots of A are the n solutions $\lambda_1 ... \lambda_n$ to the equation

$$AY = \lambda Y$$

where each element of λ is a scalar and each Y is an eigenvector. There will generally be n different eigenvalues $\lambda_1 ... \lambda_n$ and n different eigenvectors $Y_1 ... Y_n$, each corresponding to a particular eigenvalue.

Suppose $n = 2$. Then the two eigenvectors Y^a and Y^b are straight lines through the origin in the two dimensional Cartesian plane (Y_1, Y_2) such that if the initial state of the system $Y_t = (y_{1,t}, y_{2,t})$ lies on one the eigenvectors (say Y^a) then the state next period is (by definition) given by

$$Y_{t+1} = AY_t = \lambda^a Y_t$$

Now, $AY = \lambda Y$ implies $(A - \lambda I)Y = 0$. This has a non-zero solution only if $(A - \lambda I)$ is singular or has a zero determinant (one column of $A - \lambda I$ is a scalar multiple of the other).

Thus the eigenvalues can be computed by solving the polynomial

$$|A - \lambda I| = 0$$

Of course, there is a gauss command that will return all the eigenvectors and eigenvalues of any matrix, so we do not really need to worry about how to solve for them. The command is *eigv*.

The eigenvalues of a system determine the set of initial conditions, if any, under which the system is stable (see below). For example, suppose in a two dimensional system, one eigenvalue is greater than one in absolute value, and one is less than one. This means that the equilibrium exhibits a saddle-path property; the system will be stable for initial conditions that begin on the eigenvector associated with the stable eigenvalue.

1.4. Diagonalization: a nonstochastic example.. Suppose Y_t has dimension $n \times 1$.

$$Y_t = b + AY_{t-1}$$

Create an $n \times n$ matrix (Q) containing the n eigenvectors of A.

Create another $n \times n$ matrix Λ containing the n eigenvalues of A on the diagonal.

$$Q = \begin{bmatrix} Y_{1(1)} & Y_{2(1)} & ... & Y_{n(1)} \\ Y_{1(2)} & Y_{2(2)} & & Y_{n(2)} \\ . & . & . & . \\ Y_{1(n)} & Y_{2(n)} & & Y_{n(n)} \end{bmatrix} \text{ and } \Lambda = \begin{bmatrix} \lambda_1 & 0 & ... & 0 \\ 0 & \lambda_2 & & 0 \\ . & . & . & . \\ 0 & 0 & & \lambda_n \end{bmatrix}.$$

By definition

$$AQ = Q\Lambda$$

Once we have computed Q and Λ take the equation

$$Y_t = b + AY_{t-1}$$

and pre-multiply by Q^{-1}

$$Q^{-1}Y_t = Q^{-1}b + Q^{-1}AY_{t-1}$$

which can be more usefully rewritten as

$$Q^{-1}Y_t = Q^{-1}b + \Lambda Q^{-1}Y_{t-1}$$

Let $Z_t = Q^{-1}Y_t$.

$$Z_t = Q^{-1}b + \Lambda Z_{t-1}$$

The beauty of this description of the system is that the n equations are now all independent since Λ is a diagonal matrix.

1.5. The stochastic optimal growth / real business cycle model.

$$\max_{\{c_t\}} E_t \left[\sum_{t=0}^{\infty} \beta^t u(c_t) \right]$$

subject to

$$c_t + k_{t+1} \leq (1-\delta)k_t + e^{z_t} k_t^{\alpha}$$

$$z_{t+1} = \rho z_t + \varepsilon_{t+1}$$

Suppose that innovations to the productivity process (ε_{t+1}) are *iid* with mean 0 and variance σ^2.

1. Write down the first order conditions, putting variables dated $t+1$ on the left hand side (except the innovation term ε_{t+1})

$$\beta E_t \left[u'(c_{t+1}) \left(1 - \delta + \alpha e^{z_{t+1}} k_{t+1}^{\alpha-1} \right) \right] = u'(c_t)$$

$$k_{t+1} = (1-\delta)k_t + e^{z_t} k_t^{\alpha} - c_t$$

$$z_{t+1} = \rho z_t + \varepsilon_{t+1}$$

2. Solve for the steady state with $\sigma^2 = 0$

$$\beta \left[u'(c^*) \left(1 - \delta + \alpha e^{z^*} k^{*\alpha-1} \right) \right] = u'(c^*)$$

$$k^* = (1-\delta)k^* + e^{z^*} k^{*\alpha} - c^*$$

$$z^* = \rho z^*$$

These are three non-linear equations in three unknowns, k^*, c^* and z^*.

3. Take a first order Taylor series approximation of our three equations around the steady state vector $x^* = \{c^*, k^*, z^*\}$. Recall that (for x close to x^*)

$$F(x) \approx F(x^*) + DF(x^*)(x - x^*)$$

Depending on functional forms it may be possible to compute partial derivatives analytically: if it is not possible there is no harm doing in numerically - the gauss command is *gradp*.

4. Let ψ_{ij} denote the partial derivative evaluated at the steady state of the left hand side of the i^{th} equation with respect to j where $j = c, k$ or z, and ϕ_{ij} denote the same thing for the right hand side. For any variable y, let $\widehat{y} = y_t - y^*$. To a first order approximation we can write

$$E_t\left[\psi_{1c}\widehat{c}_{t+1} + \psi_{1k}\widehat{k}_{t+1} + \psi_{1z}\widehat{z}_{t+1}\right] = \phi_{1c}\widehat{c}_t + \phi_{1k}\widehat{k}_t + \phi_{1z}\widehat{z}_t$$

$$\psi_{2c}\widehat{c}_{t+1} + \psi_{2k}\widehat{k}_{t+1} + \psi_{2z}\widehat{z}_{t+1} = \phi_{2c}\widehat{c}_t + \phi_{2k}\widehat{k}_t + \phi_{2z}\widehat{z}_t$$

$$\psi_{3c}\widehat{c}_{t+1} + \psi_{3k}\widehat{k}_{t+1} + \psi_{3z}\widehat{z}_{t+1} = \phi_{3c}\widehat{c}_t + \phi_{3k}\widehat{k}_t + \phi_{3z}\widehat{z}_t \ + \varepsilon_{t+1}$$

In this particular example, a lot of the partial derivatives are zero, and others are easy to compute analytically.

$$\psi = \begin{pmatrix} \beta u''(c^*)\left(1 - \delta + \alpha e^{z^*}k^{*(\alpha-1)}\right) & \beta u'(c^*)\alpha\left(\alpha - 1\right)e^{z^*}k^{*(\alpha-2)} & \beta u'(c^*)\alpha e^{z^*}k^{*(\alpha-1)} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and

$$\phi = \begin{pmatrix} u''(c^*) & 0 & 0 \\ -1 & (1-\delta) + \alpha e^{z^*}k^{*\alpha-1} & e^{z^*}k^{*\alpha} \\ 0 & 0 & \rho \end{pmatrix}.$$

(note that $\widehat{\varepsilon}_{t+1} = \varepsilon_{t+1}$ since $\varepsilon^* = 0$).

1. Note that

$$\begin{aligned} E_t\left[\psi\widehat{x}_{t+1}\right] &= \psi E_t\left[\widehat{x}_{t+1}\right] = \psi\widehat{x}_{t+1} + \psi\left(E_t\left[\widehat{x}_{t+1}\right] - \widehat{x}_{t+1}\right) \\ &= \psi\widehat{x}_{t+1} + \psi\widehat{\omega}_{t+1} \end{aligned}$$

Thus we can write the linearized system of equations as

$$\psi\widehat{x}_{t+1} + J\widehat{\omega}_{t+1} = \phi\widehat{x}_t + \widehat{r}_{t+1}$$

where

$$\widehat{\omega}_{t+1} = \begin{pmatrix} \widehat{\omega_c} \\ \omega_k \\ \omega_z \end{pmatrix}_{t+1}, \ \widehat{r}_{t+1} = \begin{pmatrix} \widehat{0} \\ 0 \\ \varepsilon \end{pmatrix}_{t+1}, \ J = \begin{pmatrix} \psi_{11} & \psi_{12} & \psi_{13} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

or equivalently

$$\psi\widehat{x}_{t+1} = \phi\widehat{x}_t + \widehat{f}_{t+1}$$

where

$$\widehat{f}_{t+1} = \begin{pmatrix} -\psi_1\widehat{\omega}_{t+1} \\ 0 \\ \widehat{\varepsilon}_{t+1} \end{pmatrix}$$

1. Invert ψ, and premultiply through by ψ^{-1}. (For some systems ψ will not be invertible - see next week's notes).

$$\widehat{x}_{t+1} = \psi^{-1}\phi\widehat{x}_t + \psi^{-1}\widehat{f}_{t+1}$$

Let $A = \psi^{-1}\phi$.

$$\widehat{x}_{t+1} = A\widehat{x}_t + \psi^{-1}\widehat{f}_{t+1}$$

2. Compute the eigenvectors and eigenvalues of A and stack them up in a matrices Q and Λ. As before,

$$AQ = Q\Lambda$$

3. Invert Q and pre-multiply by Q^{-1}.

$$Q^{-1}\widehat{x}_{t+1} = \Lambda Q^{-1}\widehat{x}_t + Q^{-1}\psi^{-1}\widehat{f}_{t+1}$$

Let $\widehat{y}_t = Q^{-1}\widehat{x}_t$

$$\widehat{y}_{t+1} = \Lambda\widehat{y}_t + Q^{-1}\psi^{-1}\widehat{f}_{t+1}$$

Now we have successfully disentangled the system so that we can write the three equations independently. Let $\eta = Q^{-1}\psi^{-1}$. Each equation in the system now has the form

$$\widehat{y}_{it+1} = \lambda_i\widehat{y}_{it} + \eta_{row(i)}\widehat{f}_{t+1}$$

4. Reduce the dimension of the system by substituting out for one of the unknowns. Typically a system of equations of this form will have as many **unstable roots** (eigenvalues outside the unit circle) as there are **non-predetermined jump variables**; ie the system will have a saddlepoint property.

$$\begin{aligned} \#\text{ unstable roots } (|\lambda_i| > 1) &= \#\text{ jump (non-predetermined varibles) } eg \text{ consumption} \\ \#\text{ stable roots } (|\lambda_i| < 1) &= \#\text{ predetermined varibles } eg \text{ capital} \end{aligned}$$

If there are too many stable roots there will exist many rational expectations equilibria - this scenario is the basis of the literature on sunspots. If there are too few stable roots, a rational expectations equilibrium will not exist. We can determine which root in the system is unstable by examining the matrix Λ. Take conditional expectations of these equations given information at time t

$$E_t\left[\widehat{y}_{it+1}\right] = \lambda_i\widehat{y}_{it} + E_t\left[\eta_{row(i)}\widehat{f}_{t+1}\right]$$

Now the conditional forecast errors $\left(E_t\left[\widehat{f}_{t+1}\right]\right)$ are zero (by definition). Thus

$$\frac{1}{\lambda_i}E_t\left[\widehat{y}_{it+1}\right] = \widehat{y}_{it}$$

Iterating forward

$$\frac{1}{\lambda_i}E_t\left[\frac{1}{\lambda_i}E_{t+1}\left[\widehat{y}_{it+2}\right]\right] = \widehat{y}_{it}$$

and (using the law of iterated expectations)

$$\frac{1}{\lambda_i^T}E_t\left[\widehat{y}_{it+T}\right] = \widehat{y}_{it}$$

Now suppose we impose a non-explosiveness condition of the form $E_t\left[\widehat{y}_{it+T}\right] \to 0$ as $T \to \infty$. If $|\lambda_i| > 1$ (in which case we will say that the ith root is unstable or explosive), this condition implies that the only solution to the equation above is $\widehat{y}_{it} = 0$ for all t. Suppose we find that $|\lambda_i| > 1$ and thus that $\widehat{y}_{it} = 0$. Then

$$\widehat{y}_{it} = \sum_j (Q^{-1})_{ij}\widehat{x}_{jt} = (Q^{-1})_{i1}\widehat{c}_t + (Q^{-1})_{i2}\widehat{k}_t + (Q^{-1})_{i3}\widehat{z}_t = 0$$

In this case we can, for example, solve for $\widehat{c}_t$ as a linear function of $\widehat{k}_t$ and $\widehat{z}_t$.

$$\widehat{c}_t = \begin{pmatrix} c_k & c_z \end{pmatrix} \begin{pmatrix} \widehat{k_t} \\ z_t \end{pmatrix}$$

where $c_k = -\frac{(Q^{-1})_{i2}}{(Q^{-1})_{i1}}$ and $c_z = -\frac{(Q^{-1})_{i3}}{(Q^{-1})_{i1}}$. In this example therefore, the one unstable root places one linear restriction on $\widehat{x}_t$ that relates the value of the free variable, $\widehat{c}_t$ to the values of the predetermined variables $\widehat{k}_t$ and $\widehat{z}_t$. In general, for each eigenvalue in a system we find that is larger than one in absolute value, we can solve out for one variable.

5. For the unstable root $\widehat{y}_{it} = 0$ for all t. Thus

$$\widehat{y}_{it+1} = \lambda_i \widehat{y}_{it} + \eta_{row(i)}\widehat{f}_{t+1}$$

implies that

$$\eta_{row(i)}\widehat{f}_{t+1} = \eta_{row(i)} \begin{pmatrix} -\psi_1\widehat{\omega}_{t+1} \\ 0 \\ \widehat{\varepsilon}_{t+1} \end{pmatrix} = 0$$

This means that there is no room for expectational errors to have an independent effect on the equilibrium path; expectational errors in the intertemporal Euler equation are linearly related to innovations to productivity:

$$\psi_1\widehat{\omega}_{t+1} = \frac{\eta_{i,3}}{\eta_{i,1}}\widehat{\varepsilon}_{t+1}$$

6. Simply plug the expression for $\widehat{c}_t$ derived in Step 9 back into the system, and reduce the number of equations and variables by 1, ending up with a system in just $\widehat{k}$ and $\widehat{z}$.

$$\begin{aligned}\begin{pmatrix}\widehat{k}\\ z\end{pmatrix}_{t+1} &= \left(\begin{pmatrix}A_{21}\\ A_{31}\end{pmatrix}\begin{pmatrix}c_k & c_z\end{pmatrix}+\begin{pmatrix}A_{22} & A_{23}\\ A_{32} & A_{33}\end{pmatrix}\right)\begin{pmatrix}\widehat{k}\\ z\end{pmatrix}_t+\begin{pmatrix}\psi^{-1}_{row(2)}\\ \psi^{-1}_{row(3)}\end{pmatrix}\begin{pmatrix}-\frac{\eta_{i,3}}{\eta_{i,1}}\widehat{\varepsilon}_{t+1}\\ 0\\ \widehat{\varepsilon}_{t+1}\end{pmatrix}\\ &= \left(\begin{pmatrix}A_{21}\\ A_{31}\end{pmatrix}\begin{pmatrix}c_k & c_z\end{pmatrix}+\begin{pmatrix}A_{22} & A_{23}\\ A_{32} & A_{33}\end{pmatrix}\right)\begin{pmatrix}\widehat{k}\\ z\end{pmatrix}_t+\begin{pmatrix}\widehat{0}\\ \varepsilon\end{pmatrix}_{t+1}\end{aligned}$$

2. SUNSPOTS

Suppose, for the sake of expositional simplicity, that there is no fundamental uncertainty in the economy, i.e. that $\sigma^2 = 0$. This means that we can eliminate one equation and one unknown (z) from our system. Thus we now have

$$\psi\widehat{x}_{t+1} + J\widehat{\omega}_{t+1} = \phi\widehat{x}_t + \widehat{r}_{t+1}$$

where

$$\widehat{\omega}_{t+1} = E_t\left[\widehat{x}_{t+1}\right] - \widehat{x}_{t+1} = \begin{pmatrix}\widehat{\omega_c}\\ \omega_k\end{pmatrix}_{t+1}, \widehat{r}_{t+1} = \begin{pmatrix}\widehat{0}\\ 0\end{pmatrix}_{t+1}, J = \begin{pmatrix}\psi_{11} & \psi_{12}\\ 0 & 0\end{pmatrix}$$

This can be written as

$$\psi\widehat{x}_{t+1} = \phi\widehat{x}_t + \widehat{f}_{t+1}$$

where

$$\widehat{f}_{t+1} = \begin{pmatrix}-\psi_1\widehat{\omega}_{t+1}\\ 0\end{pmatrix} = \begin{pmatrix}\varepsilon_{t+1}\\ 0\end{pmatrix}$$

Inverting ψ (assuming this is possible)

$$\widehat{x}_{t+1} = \psi^{-1}\phi\widehat{x}_t + \psi^{-1}\widehat{f}_{t+1}$$

Compute the eigenvalues of $A = \psi^{-1}\phi$.

Now suppose (hypothetically) that A has 2 eigenvalues inside the unit circle (ie the system has two stable roots). Let ε_{t+1} be any random variable that is unforecastable at date t in that $E_t(\varepsilon_{t+1}) = 0$.

Suppose $\widehat{c}_1 = \widehat{k}_1 = 0$ (we are in the non-stochastic steady state)

The equation above defines a stable Markov process.

Thus the model generates business cycles driven by a non-fundamental random variable; even with no 'real' shock at $t+1$ (since there is no fundamental uncertainty) the realized value for x_{t+1} can differ from the expected value at t. Note that since the only requirement for the non-fundamental shock process is that $E_t(\varepsilon_{t+1}) = 0$ this economy has an infinite number of equilibria.

3. THE SCHUR DECOMPOSITION (SEE KLEIN 2000 AND SODERLIND 1999)

In some cases once we have linearized the system of equations characterizing equilibrium and have it in the form

$$\psi \widehat{x}_{t+1} = \phi \widehat{x}_t + \widehat{f}_{t+1}$$

the matrix ψ turns out to be non-invertible. This will happen if we have some non-intertemporal equations in our system, such as a first order condition for labor supply. In this case we need to use slightly more sophisticated linear algebra.

1. Decide what variables are going to be in the system of equations and solve for the steady state (balanced growth path). Let x denote the vector of variables in the system.

2. Order the variables and equations characterizing equilibrium so that:

   (a) Date $t+1$ variables (except innovations to shocks) are on the LHS and date t variables are on the RHS.

   (b) Predetermined variables (eg values for shocks, capital stocks) come first and non-pre-determined (eg labor supply) variables second.

   (c) Inter-temporal equations (eg inter-temporal budget constraints, laws of motion for shocks, inter-temporal Euler equations) are placed above, and intra-temporal equations below.

   (d) The inter-temporal Euler equations are placed last among the set of inter-temporal equations.

3. Take first order Taylor series approximations around the steady state values for state variables. Let ψ_{ij} denote the partial derivative evaluated at the steady state of the left hand side (LHS) of the (expectation of the) i^{th} equation with respect to j and ϕ_{ij} denote the same thing for the right hand side (RHS).

4. Stack up the linearized system of equations in a matrix system following the ordering described above.

   $$\psi \widehat{x}_{t+1} + J\widehat{\omega}_{t+1} = \phi \widehat{x}_t + \widehat{v}_{t+1}$$

   where $\widehat{\omega}_{t+1} = E_t\left[\widehat{x}_{t+1}\right] - \widehat{x}_{t+1}$. Let k denote the row(s) corresponding to the inter-temporal Euler equations. For $h \neq k$, expectation terms do not enter equations and thus the corresponding row of J is all zeros. The k^{th} row of J is ψ_k. The LHS of the equations corresponding to the inter-temporal Euler

equations are $\psi_k \widehat{x}_{t+1} + \psi_k \widehat{\omega}_{t+1} = \psi_k E_t [\widehat{x}_{t+1}] = E_t [\psi_k \widehat{x}_{t+1}]$ .Let l denote the row(s) corresponding to the laws of motion for the exogenous shocks. For $h \neq l$, $\widehat{v}_{ht+1} = 0$. Let $\widehat{v}_{lt+1} = s_{t+1}$. Now this system is clearly equivalent to a system with errors in equations given by $\widehat{f}_{t+1}$ where

$$\psi \widehat{x}_{t+1} = \phi x_t + \widehat{f}_{t+1}$$

and

$$\widehat{f}_{it+1} = \begin{cases} s_{t+1} & i = l \\ -\psi_k \omega & i = k \\ 0 & i \neq k, l \end{cases}$$

5. Perform the Generalized Complex Schur Decomposition. Given ψ and ϕ this returns Q, Z, S and T such that

$$\psi = QSZ^H$$

$$\phi = QTZ^H$$

$$QSZ^H \widehat{x}_{t+1} = QTZ^H x_t + \widehat{f}_{t+1}$$

Here S and T are upper triangular and Q and Z are orthogonal. The latter property means that for example $Q^H Q = I$ where Q^H is the transpose of the complex conjugate of Q. The decomposition should be ordered so that the stable generalized eigenvalues come first where the generalized eigenvalue is the i^{th} element of T divided by the i^{th} element of S (this will either be done automatically or will be an option you can set when invoking the command). Using Paul Soderlind's Gauss routine, (available at http://www.hhs.se/personal/PSoderlind/) the Decomposition may be called as follows.

$$\{T, S, Q, Z, \lambda^G\} = zTgSen(\phi, \psi, cutoff, 1, 0)$$

Here λ^G are the 'generalized eigenvalues'. The $cutoff$ argument is used to tell Gauss what should count as a stable eigenvalue and what as an unstable one - if the absolute value of the i^{th} generalized eigenvalue λ_i^G is less than $cutoff$ then this eigenvalue is stable. In most cases choosing $cutoff = 1$ should be appropriate. The final two arguments determine ordering.

6. Count the number of stable eigenvalues - call this n_s. Let $n_u = n_{var} - n_s$ denote the number of unstable eigenvalues (n_{var} being the number of variables in the system).

7. Premultiply through by Q^H.

$$SZ^H \widehat{x}_{t+1} = TZ^H x_t + Q^H \widehat{f}_{t+1}$$

8. Let $\left(\begin{array}{c}\widehat{s}\\ u\end{array}\right)_t$ be defined by $Z\left(\begin{array}{c}\widehat{s}\\ u\end{array}\right)_t = x_t$ where the partition is such that s has dimension $n_s \times 1$, and u is $n_u \times 1$. This implies that

$$\left(\begin{array}{c}\widehat{s}\\ u\end{array}\right)_t = Z^H x_t \tag{5}$$

and thus we can write

$$S\left(\begin{array}{c}\widehat{s}\\ u\end{array}\right)_{t+1} = T\left(\begin{array}{c}\widehat{s}\\ u\end{array}\right)_t + Q^H \widehat{f}_{t+1}$$

9. Partitition the columns of S, T, Z and Q^H conformably with s and u. Thus, for example, S_{11} is of dimension $n_s \times n_s$, while S_{12} is $n_s \times n_u$.

$$\left(\begin{array}{cc}S_{11} & S_{12}\\ S_{21} & S_{22}\end{array}\right)\left(\begin{array}{c}\widehat{s}\\ u\end{array}\right)_{t+1} = \left(\begin{array}{cc}T_{11} & T_{12}\\ T_{21} & T_{22}\end{array}\right)\left(\begin{array}{c}\widehat{s}\\ u\end{array}\right)_t + \left(\begin{array}{cc}Q_{11}^H & Q_{12}^H\\ Q_{21}^H & Q_{22}^H\end{array}\right)\left(\begin{array}{c}\widehat{f_{st+1}}\\ f_{ut+1}\end{array}\right)$$

10. Since S_{21} and T_{21} are zeros, and the generalized eigenvalues corresponding to the last n_u equations are unstable, the only stable solution to this system of equations requires that $\widehat{u}_t = 0$ for all t. The first set of equations are then

$$S_{11}\widehat{s}_{t+1} = T_{11}\widehat{s}_t + \left(Q_{11}^H \; Q_{12}^H\right)\left(\begin{array}{c}\widehat{f_{st+1}}\\ f_{ut+1}\end{array}\right) \tag{6}$$

The second set of equations are

$$\left(Q_{21}^H \; Q_{22}^H\right)\left(\begin{array}{c}\widehat{f_{st+1}}\\ f_{ut+1}\end{array}\right) = 0 \tag{7}$$

which implies that (assuming that Q_{22}^H is invertible)

$$\widehat{f}_{ut+1} = -\left(Q_{22}^H\right)^{-1} Q_{21}^H \widehat{f}_{st+1} \tag{8}$$

11. Note that 5 implies that

$$\left(\begin{array}{c}\widehat{x_s}\\ x_u\end{array}\right)_t = \left(\begin{array}{c}Z_{11}\\ Z_{21}\end{array}\right)\widehat{s}_t \qquad \left[+\left(\begin{array}{c}Z_{21}\\ Z_{22}\end{array}\right)\widehat{u}_t\right]$$

Supposing Z_{11} is invertible, we have

$$\widehat{s}_t = Z_{11}^{-1}\widehat{x}_{st}$$

and

$$x_{ut} = Z_{21}\widehat{s}_t = Z_{21}Z_{11}^{-1}\widehat{x}_{st}$$

12. Substitute $\widehat{s}_t = Z_{11}^{-1}\widehat{x}_{st}$ into 6

$$S_{11}Z_{11}^{-1}\widehat{x}_{st+1} = T_{11}Z_{11}^{-1}\widehat{x}_{st} + \begin{pmatrix} Q_{11}^H & Q_{12}^H \end{pmatrix} \begin{pmatrix} \widehat{f_{st+1}} \\ f_{ut+1} \end{pmatrix}$$

13. Premultiply through by S_{11}^{-1} and substitute in 8.

$$Z_{11}^{-1}\widehat{x}_{st+1} = S_{11}^{-1}T_{11}Z_{11}^{-1}\widehat{x}_{st} + S_{11}^{-1}\begin{pmatrix} Q_{11}^H & Q_{12}^H \end{pmatrix} \begin{pmatrix} \widehat{f}_{st+1} \\ -\left(Q_{22}^H\right)^{-1}Q_{21}^H\widehat{f}_{st+1} \end{pmatrix}$$

$$\widehat{x}_{st+1} = Z_{11}S_{11}^{-1}T_{11}Z_{11}^{-1}\widehat{x}_{st} + Z_{11}S_{11}^{-1}\left(Q_{11}^H - Q_{12}^H\left(Q_{22}^H\right)^{-1}Q_{21}^H\right)\widehat{f}_{st+1}$$