Stochastic income-fluctuations problem

Focus on the case where shocks are iid through time

Look at the cases $\beta(1+r) = 1$ and $\beta(1+r) > 1$

End up looking at the two shock case $e \in \{e_l, e_h\}$ with constant relative risk aversion preferences

Assume we have made sufficient assumptions to ensure

- principle of optimality applies

- can apply the contraction mapping theorem

- value function v is strictly increasing (in all arguments), strictly concave

The Recursive formulation of this problem is

$$V(a,e) = \max_{c,a'} \left\{ u(c) + \beta \sum_{e' \in \{e_l, e_h\}} \pi(e') V(a', e') \right\}$$

subject to

$$\begin{aligned} c + a' &= e + (1+r)a \\ a' &\geq -\phi \\ c &\geq 0 \end{aligned}$$

If productivity shocks are i.i.d we can take a transformation of the state variables to eliminate e as a state variable.

Let

$$z = e + (1 + r)a + \phi$$

Thus z denotes maximum disposable resources (maximum possible consumption given e and a, the interest rate r, and the borrowing constraint ϕ). In terms of the single state variable z the household's budget set is given by

$$c + a' \leq z - \phi$$

If the probability distribution over e' is independent of e, then provided the household knows z he does not need to know e to solve his optimization problem. In other words, the household does not care whether the resources he has at his disposal come from savings he made in the previous period or from his current period endowment shock.

We can therefore rewrite the value function as follows.

$$V(z) = \max_{c,a'} \left\{ u(c) + \beta \sum_{e' \in \{e_l, e_h\}} \pi(e') V(z') \right\}$$

$$c = z - \phi - a'$$

$$z' = e' + (1 + r)a' + \phi$$

$$a' \geq -\phi$$

Denote the decision rules that solve this problem $c(z)$ and $a'(z)$.

The first order condition is

$$u'(c) \geq \beta(1+r) \sum_{e' \in \{e_l, e_h\}} \pi(e') V'(z') \qquad = \text{if } a' > -\phi \tag{1}$$

Note that the transition probabilities do not depend on e. The envelope condition here is

$$V'(z) = u'(c) \tag{2}$$

(The first order condition must be as before, since we have redefined the state variable, without changing the underlying problem)

1 Result 1 (K Props 102/104)

(1) Consumption is strictly increasing in z : $\frac{dc}{dz} > 0$

(2) If there is a value for z, $\underline{z}$, such that $a'(\underline{z}) = -\phi$ then (1) $a'(z) = -\phi$ for all $z \leq \underline{z}$, and (2) if $a'(\tilde{z}) > -\phi$ then $\frac{da'}{dz} > 0$ for all $z \geq \tilde{z}$

(3) $\frac{dc}{dz} \leq 1$ and $\frac{da'}{dz} < 1$

Proofs:

(1) Differentiate the envelope condition

$$V''(z) = u''(c)\frac{dc}{dz}$$

Since V and u are strictly concave, $\frac{dc}{dz} > 0$

(2) (second part) Suppose the borrowing constraint is not binding: $a'(z) > -\phi$

The first order condition is

$$u'(c) = \beta(1+r) \sum_{e' \in \{e_l, e_h\}} \pi(e') V'(e' + (1+r)a' + \phi)$$

Differentiating with respect to z

$$u''(c)\frac{dc}{dz} = \beta(1+r)^2 \sum_{e' \in \{e_l, e_h\}} \pi(e') V''(z') \frac{da'}{dz}$$

so $\frac{da'}{dz} > 0$.

(2) (first part) Suppose $a'(\underline{z}) = -\phi$, $z < \underline{z}$ and (contrary to the proposition) $a'(z) > -\phi$. We will derive a contradiction.

Because V is concave, for any e_i

$$\beta V'(e_i + (1+r)a'(z) + \phi) < \beta V'(e_i + (1+r)a'(\underline{z}) + \phi)$$

(the argument on the right hand side is smaller than the argument on the left hand side)

So

$$\begin{aligned} u'(c(z)) &= \beta \sum_{e' \in \{e_l, e_h\}} \pi(e') V'(e' + (1+r)a'(z) + \phi) \Rightarrow \\ u'(c(z)) &< \beta \sum_{e' \in \{e_l, e_h\}} \pi(e') V'(e' + (1+r)a'(\underline{z}) + \phi) \end{aligned}$$

But we have already proven that

$$c(\underline{z}) > c(z) \Leftrightarrow u'(c(\underline{z})) < u'(c(z))$$

Combining we get that

$$\begin{aligned} u'(c(z)) &> \underbrace{u'(c(\underline{z})) \geq \beta(1+r) \sum_{e' \in \{e_l, e_h\}} \pi(e') V'(e' + (1+r)a'(\underline{z}) + \phi)}_{\text{original FOC}} \\ &> \beta(1+r) \sum_{e' \in \{e_l, e_h\}} \pi(e') V'(e' + (1+r)a'(z) + \phi) = u'(c(z)) \end{aligned}$$

Contradiction..

(Question we will come back to: how can we be sure that $a'(z) > -\phi$ is not optimal for any $z \geq z_{\text{min}} = e_l - (1+r)\phi + \phi$?)

(3) We know that

$$c(z) + a'(z) = z - \phi$$

Differentiating in the region $z > \underline{z}$ gives

$$\frac{dc}{dz} + \frac{da'}{dz} = 1$$

We have already show that both functions are increasing, so it follows that $\frac{dc}{dz} < 1$ and $\frac{da'}{dz} < 1$.

2 Result 2

In this case we will prove that if $\beta(1+r) \geq 1$ there is no upper bound on asset holding. This goes some way towards showing that consumption does not converge to a finite constant (without relying on any assumptions on the convexity of $u'(c)$)

Showing that assets are unbounded above is equivalent to showing that z is unbounded above, given that e is bounded and r and ϕ are finite constants.

Total resources will remain bounded through time if there exists a z, denoted z_{max}, such that for all e' and for all $z \leq z_{\text{max}}$

$$z' = e' + (1+r)a'(z) + \phi \leq z_{\text{max}}$$

Note that z' is increasing in next period's endowment e'. Note also from Result 1 that $\frac{da'}{dz} \geq 0$, so z' is also increasing in z.

So it suffices to check that

$$e_h + (1+r)a'(z_{\text{max}}) + \phi \leq z_{\text{max}}$$

If we can find such a finite z_{max} then provided that we start out with $z_0 \leq z_{\text{max}}$, disposable resources will never exceed z_{max}.

Showing that $\beta(1+r) < 1$ is necessary for boundedness

Let us assume that there exists a z_{max} as defined above, and that that $\beta(1+r) \geq 1$. We will then derive a contradiction. This will tell us that for any possible candidate for z_{max}, there is a positive probability that resources will increase beyond z_{max}.

Substituting the envelope condition 2 into the first order condition 1 we get

$$V'(z) \geq \beta(1+r) \sum_{e'} \pi(e') V'(z')$$

If $z = z_{\text{max}}$ this implies that

$$V'(z_{\text{max}}) \geq \beta(1+r) \sum_{e'} \pi(e') V'(e' + (1+r)a'(z_{\text{max}}) + \phi)$$

Since V is strictly concave, V' is decreasing

$$\sum_{e'} \pi(e') V'(e' + (1+r)a'(z_{\mathsf{max}}) + \phi) > V'(e_h + (1+r)a'(z_{\mathsf{max}}) + \phi) \geq V'(z_{\mathsf{max}})$$

Thus we have shown that

$$V'(z_{\mathsf{max}}) > \beta(1+r)V'(z_{\mathsf{max}})$$

Now if $\beta(1+r) \geq 1$ this is a contradiction, which means that there does not exist a z_{max} as defined. This suggests that z_t will diverge to $+\infty$.

Thus we have shown that $\beta(1+r) < 1$ is a necessary condition for asset holdings to remain bounded.

Is $\beta(1+r) < 1$ sufficient for boundedness? Can we show that the state variable z is bounded for $\beta(1+r) < 1$?

First it is straightforward to show that z is bounded below, since

$$z \geq e_l - (1+r)\phi + \phi = e_l - r\phi$$

3 Result 3 (K Prop 105)

If $\beta(1+r) < 1$ then there is a value $\overline{z} > z_{\text{min}} = e_l - r\phi$ such that for all $z \le \overline{z}$, $c(z) = z$ and $a'(z) = -\phi$

Proof: Suppose not. Then for all $z \ge e_l - r\phi$, $a'(z) > -\phi$ and thus

$$\begin{aligned} V'(z) &= \beta(1+r) \sum_{e' \in \{e_l, e_h\}} \pi(e') V'(e' + (1+r)a'(z) + \phi) \\ &\le \beta(1+r) V'(e_l + (1+r)a'(z) + \phi) < V'(e_l + (1+r)a'(z) + \phi) \end{aligned}$$

Pick $z = e_l - r\phi$. Then the argument of the rightmost term is larger than the argument of the leftmost term, contradictin the inequality.

Thus there is a cut-off level for available resources $\overline{z}$ such that if the consumer has less that $\overline{z}$ he consumes $\overline{z}$ and saves $-\phi$.

4 Result 4 (K Prop 106, Schechtman and Escudero 1977)

If (i) the utility function is of the form $u(c) = -e^{-c}$, (ii) $\phi = 0$, and (iii) $y_l = 0$ then $z \rightarrow \infty$ almost surely, even if $\beta(1 + r) < 1$.

5 Result 5 (K Prop 107, Schechtman and Escudero 1977)

If there exists a finite value μ such that

$$\lim_{c \to \infty} \left(\log_c u'(c) \right) = \mu$$

then there exists an upper bound on z, z_{max} as defined previously.

Suppose

$$u(c) = \frac{1}{1-\sigma} c^{1-\sigma}$$

$$\log_c(u'(c)) = \log_c(c^{-\sigma}) = -\sigma \log_c(c) = -\sigma$$

so the theorem applies.

6 Result 6 (Huggett 1993)

In the two shock case, with CRRA preferences and $\beta(1+r) < 1$ there exists an upper bound on asset holdings even with persistent shocks, as long as $\pi(e_h|e_h) \geq \pi(e_h|e_l)$.