Consider the following economy

Each period a continuum of mass $(1-\delta)$ agents are born at age 0

Agents survive from age a to $a+1$ with constant probability δ.

The total population is

$$(1-\delta)(1+\delta+\delta^2+...)=1$$

Assume the environment is stationary, so we need not worry about time subscripts

Agents maximize expected utility, which is given by

$$E\sum_{a=0}^{\infty}(\beta\delta)^a u(c_a)$$

where

$$u(c)=\frac{c^{1-\gamma}}{1-\gamma}$$

Agents are subject to idiosyncratic wage shocks that are *iid* across agents. Agents supply one unit of time per period. The initial wage at age zero is lognormally distributed:

$$\ln(w_0)=\alpha_0 \sim N\left(-\frac{v_0}{2},v_0\right)$$

Wages subsequently evolve according to

$$\begin{aligned}\alpha_{a+1} &= \alpha_a+\omega_{a+1} \qquad a\geq 0\\ \omega_{a+1} &\sim N\left(\mu-\frac{v_\omega}{2},v_\omega\right)\end{aligned}$$

where the actual (level) wage is $w_a=\exp(\alpha_a)$.

The market structure is as follows. Agents are endowed with zero wealth at birth. Then they can trade bonds at a constant price q. They cannot trade prior to drawing α_0. The generic budget constraint is

$$\begin{aligned}c_a+qb_{a+1} &= b_a+w_a \qquad a\geq 0\\ b_0 &= 0\end{aligned}$$

There is a borrowing constraint,

$$b_{a+1}\geq\phi$$

Bonds are in zero net supply.

1. Suppose, to start that $\phi=0$. Argue that in equilibrium it must be the case that

$$\begin{aligned}b_a &= 0 \qquad a\geq 0\\ c_a &= w_a \qquad a\geq 0\end{aligned}$$

Argue (intuitively) that any value for q above a certain threshold is an equilibrium.

Agents cannot borrow by assumption. They cannot save because if no agents are borrowing, aggregate savings will be positive if a postive measure of agents save, and the aggregate supply of bonds is zero. To make sure no agents want to save, all that is required is a sufficiently low interest rate.

2. Suppose now that $\phi < 0$. Show that there is an equlibrium with allocations as in part (1) and derive an expression for the (now unique) equilibrium value for q.[1]

   Now agents can potentially borrow. We want to construct an equilibrium in which agents do not do so. The inter-temporal first order condition for an agent with age a and current wage w is

$$c_a(w_a)^{-\gamma} q = \beta\delta E_a \left[c_{a+1}(w_{a+1})^{-\gamma} \right]$$

   Suppose allocations are as describe above. Then $c_a(w_a) = \exp(\alpha_a)$ and $c_{a+1}(w_{a+1}) = \exp(\alpha_a + \omega_{t+1})$,implying a bond price required to support this equilibrium of

$$\begin{aligned} q &= \frac{\beta\delta E_a \left[c_{a+1}(w_{a+1})^{-\gamma} \right]}{c_a(w_a)^{-\gamma}} = \frac{\beta\delta E_a \left[\exp(-\gamma\alpha_a) \exp(-\gamma\omega_{a+1}) \right]}{\exp(-\gamma\alpha_a)} \\ &= \beta\delta E_a \left[\exp(-\gamma\omega_{a+1}) \right] \end{aligned}$$

   Using the "trick",

$$E_a \left[\exp(-\gamma\omega_{a+1}) \right] = \exp(-\gamma\mu + \gamma\frac{v_\omega}{2} + \gamma^2 \frac{v_\omega}{2})$$

   So

$$q = \beta\delta \exp(-\gamma\mu) \exp(\gamma(\gamma+1)\frac{v_\omega}{2})$$

   This expression is independent of the individual states a and w, so we have effectively shown that our candidate allocations and price satisfy the agent's inter-temporal FOC for any possible pair (α, w).

3. We could imagine cross-sectional consumption inequality in this economy increasing because of (i) an increase in μ (which would increase inequality across age groups), (ii) because of an increase in v_0, or (iii) because of an increase in v_ω. Assuming $\gamma > 1$, characterize the effects of these three different changes on the equilibrium interest rate $r = \frac{1-q}{q}$ and give some intuition.

[1] Here is a useful trick which I tried to explain in class that will be useful for this problem. Suppose $\omega \tilde{} N(\mu, \sigma^2)$. Then $E\left[\exp(\omega)\right] = \exp(\mu + \frac{\sigma^2}{2})$. It follows directly that for some constant κ, $\kappa\omega \tilde{} N(\kappa\mu, \kappa^2\sigma^2)$, so $E\left[\exp(\kappa\omega)\right] = E\left[\exp(\omega)^\kappa\right] = \exp(\kappa\mu + \frac{\kappa^2\sigma^2}{2})$.

(i) increasing μ lowers q and raises r (intuition: with faster expected consumption growth, a higher interest rate is required to dissuade agents from borrowing)

(ii) v_0 does not affect the equilibrium interest rate since α_0 is drawn before the bond market opens

(iii) increasing v_ω raises the bond price and lowers the equilibrium interest rate (intuition: more volatile consumption growth raises demand for precautionary savings, a lower equilibrium interest cools down this extra demand).

4. Derive an expression for expected lifetime utility for a newborn agent who has yet to draw any shocks.

   Expected lifetime utility is given by

$$
\begin{aligned}
E\sum_{a=0}^{\infty}(\beta\delta)^a u(c_a) &= E\left[\frac{\exp((1-\gamma)\alpha_0)}{(1-\gamma)}\right]+\beta\delta E\left[\frac{\exp((1-\gamma)(\alpha_0+\omega_1)}{(1-\gamma)}\right] \\
&\quad +(\beta\delta)^2 E\left[\frac{\exp((1-\gamma)(\alpha_0+\omega_1+\omega_2)}{(1-\gamma)}\right]+... \\
&= \frac{1}{1-\gamma}E\left[\exp((1-\gamma)\alpha_0)\right]\left[\begin{array}{c} 1+\beta\delta E\left[\exp((1-\gamma)\omega_1\right] \\ +(\beta\delta)^2 E\left[\exp((1-\gamma)\omega_1\right]E\left[\exp((1-\gamma)\omega_2\right]+... \end{array}\right] \\
&= \frac{1}{1-\gamma}\exp(-\gamma(1-\gamma)\frac{v_0}{2})\left[1+\beta\delta\exp\left((1-\gamma)\mu-\gamma(1-\gamma)\frac{v_\omega}{2}\right)+(\beta\delta)^2\exp\left((1-\gamma)\mu-\gamma(1-\gamma)\frac{v_\omega}{2}\right.\right. \\
&= \frac{1}{1-\gamma}\exp(-\gamma(1-\gamma)\frac{v_0}{2})\frac{1}{1-\beta\delta\exp\left((1-\gamma)\mu-\gamma(1-\gamma)\frac{v_\omega}{2}\right)}
\end{aligned}
$$

5. Define the (steady state) welfare effect of a change in the wage process from (μ, v_0, v_ω) to $(\widehat{\mu}, \widehat{v}_0, \widehat{v}_\omega)$ as the value for λ that satisfies

$$
E_{|(\mu,v_0,v_\omega)}\sum_{a=0}^{\infty}(\beta\delta)^a u(c_a(1+\lambda)) = E_{|(\widehat{\mu},\widehat{v}_0,\widehat{v}_\omega)}\sum_{a=0}^{\infty}(\beta\delta)^a u(c_a)
$$

   This welfare measure basically asks "by what percentage amount would one have to increase consumption in every date and state in an economy with (μ, v_0, v_ω) to leave an agent indifferent between being born into the (μ, v_0, v_ω) economy versus being born into the $(\widehat{\mu}, \widehat{v}_0, \widehat{v}_\omega)$ economy." Assume $\beta = 0.96$, $\delta = 0.98$, $\gamma = 2$, $\mu = 0.02$, $v_0 = 0.2$, $v_\omega = 0.02$. Using your answer to (4) compute values for λ corresponding to

   (a) $\mu \to \widehat{\mu} = 0.03$

   (b) $v_0 \to \widehat{v}_0 = 0.3$

   (c) $v_\omega \to \widehat{v}_\omega = 0.03$

   Take (a). We have

$$
\exp(-\gamma(1-\gamma)\frac{v_0}{2})\frac{1}{1-\beta\delta\exp\left((1-\gamma)\mu-\gamma(1-\gamma)\frac{v_\omega}{2}\right)}(1+\lambda)^{1-\gamma} = \exp(-\gamma(1-\gamma)\frac{v_0}{2})\frac{1}{1-\beta\delta\exp\left((1-\gamma)\widehat{\mu}-\gamma(1-\gamma)\frac{v_\omega}{2}\right)}
$$

or

$$
\begin{aligned}
\frac{1}{1-\beta\delta\exp\left((1-\gamma)\mu-\gamma(1-\gamma)\frac{v_\omega}{2}\right)}(1+\lambda)^{1-\gamma} &= \exp(-\gamma(1-\gamma)\frac{\Delta v_0}{2})\frac{1}{1-\beta\delta\exp\left((1-\gamma)\widehat{\mu}-\gamma(1-\gamma)\frac{v_\omega}{2}\right)} \\
\frac{1}{1-0.96\times0.98\exp\left((1-2)0.02-2(1-2)\frac{0.02}{2}\right)}(1+\lambda)^{1-2} &= \frac{1}{1-0.96\times0.98\exp\left((1-2)0.03-2(1-2)\frac{0.02}{2}\right)}
\end{aligned}
$$

Solution is: $\lambda = 0.158$. Thus a 1% point increase in the growth rate is worth a permanent (but once and for all) 15.8 percent increase in current consumption

(b)

$$
\begin{aligned}
\frac{1}{1-\beta\delta\exp\left((1-\gamma)\mu-\gamma(1-\gamma)\frac{v_\omega}{2}\right)}(1+\lambda)^{1-\gamma} &= \exp(-\gamma(1-\gamma)\frac{\Delta v_0}{2})\frac{1}{1-\beta\delta\exp\left((1-\gamma)\widehat{\mu}-\gamma(1-\gamma)\frac{v_\omega}{2}\right)} \\
(1+\lambda)^{1-2} &= \exp(-2(1-2)\frac{0.1}{2})
\end{aligned}
$$

Solution is $\lambda = -0.095$

(c)

$$
\begin{aligned}
\frac{1}{1-\beta\delta\exp\left((1-\gamma)\mu-\gamma(1-\gamma)\frac{v_\omega}{2}\right)}(1+\lambda)^{1-\gamma} &= \exp(-\gamma(1-\gamma)\frac{\Delta v_0}{2})\frac{1}{1-\beta\delta\exp\left((1-\gamma)\widehat{\mu}-\gamma(1-\gamma)\frac{v_\omega}{2}\right)} \\
\frac{1}{1-0.96\times0.98\exp\left((1-2)0.02-2(1-2)\frac{0.02}{2}\right)}(1+\lambda)^{1-2} &= \frac{1}{1-0.96\times0.98\exp\left((1-2)0.02-2(1-2)\frac{0.03}{2}\right)}
\end{aligned}
$$

, Solution is $\lambda = -0.160$