Macroeconomics

Lecture 12: dynamic programming applications, part three

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This class

- Dynamic programming applications, part three
- Consumption-savings problems
  - review of linear-quadratic permanent income theory
  - effects of income uncertainty in more general settings

Review of permanent income theory

- Time $t = 0, 1, 2, \dots$

- Single agent with risk averse preferences

$$\mathbb{E}\left\{ \sum_{t=0}^{\infty} \beta^t u(c_t) \right\}, \qquad 0 < \beta < 1$$

- Flow budget constraint

$$a_{t+1} = R(a_t + y_t - c_t)$$

given some stochastic process for income y_t

- Consumption Euler equation

$$u'(c_t) = \beta R \mathbb{E}_t \{ u'(c_{t+1}) \}$$

Hall (1978)

- Strict version of the *permanent income hypothesis* (PIH)
  - quadratic utility
    $$u(c) = c - \frac{b}{2}c^2, \qquad b > 0$$
  - interest rate equals rate of time preference
    $$\beta R = 1$$
- Then consumption Euler equation simply implies
  $$c_t = \mathbb{E}_t\{\, c_{t+1} \,\}$$
- Implies *consumption is a martingale* (e.g., a random walk).
  More generally, marginal utility is a martingale

Iterating forward

- At $t = 0$ we have

$$a_1 = R(a_0 + y_0 - c_0)$$

- At $t = 1$ we have

$$a_2 = R(a_1 + y_1 - c_1) = R^2(a_0 + y_0 - c_0) + R(y_1 - c_1)$$

- At $t = 2$ we have

$$a_3 = R^3(a_0 + y_0 - c_0) + R^2(y_1 - c_1) + R(y_2 - c_2)$$

Iterating forward

- Iterating this out to some arbitrary date T

$$a_{T+1} = R^{T+1} a_0 + \sum_{t=0}^{T} R^{T+1-t}(y_t - c_t)$$

- Dividing both sides by R^{T+1} and rearranging

$$\sum_{t=0}^{T} R^{-t} c_t + R^{-(T+1)} a_{T+1} = a_0 + \sum_{t=0}^{T} R^{-t} y_t$$

- Taking $T \to \infty$ and imposing the no-Ponzi-scheme constraint

$$\sum_{t=0}^{\infty} R^{-t} c_t = a_0 + \sum_{t=0}^{\infty} R^{-t} y_t$$

Intertemporal budget constraint

- Nothing special about period $t = 0$ so write this as

$$\sum_{j=0}^{\infty} R^{-j} c_{t+j} = a_t + \sum_{j=0}^{\infty} R^{-j} y_{t+j}$$

- Also holds in expectation

$$\mathbb{E}_t \left\{ \sum_{j=0}^{\infty} R^{-j} c_{t+j} \right\} = a_t + \mathbb{E}_t \left\{ \sum_{j=0}^{\infty} R^{-j} y_{t+j} \right\}$$

Solving for consumption

- Interchanging the sum and expectations

$$\sum_{j=0}^{\infty} R^{-j}\mathbb{E}_t\left\{c_{t+j}\right\} = a_t + \sum_{j=0}^{\infty} R^{-j}\mathbb{E}_t\left\{y_{t+j}\right\}$$

- But from the consumption Euler equation and the law of iterated expectations

$$\mathbb{E}_t\left\{c_{t+j}\right\} = c_t \qquad \text{for all } j$$

Solving for consumption

- This gives the solution

$$c_t = \frac{r}{1+r}\left(a_t + \sum_{j=0}^{\infty} R^{-j}\mathbb{E}_t\left\{y_{t+j}\right\}\right), \qquad R = 1 + r$$

- It is customary to refer to a_t as ‘financial wealth; and to define ‘human wealth’ h_t by

$$h_t \equiv \sum_{j=0}^{\infty} R^{-j}\mathbb{E}_t\left\{y_{t+j}\right\}$$

- Consumption out of total wealth (i.e., ‘permanent income’) is

$$c_t = \frac{r}{1+r} w_t = (1-\beta) w_t \qquad w_t \equiv a_t + h_t$$

Certainty equivalence

- Solution exhibits *certainty equivalence*. Optimal c_t policy depends only on expected y_{t+j}

- Higher moments do not matter. In particular, income risk (*volatility* of y_{t+j}) does not matter for optimal c_t

- This is a because of the linear-quadratic specification

- Volatility of y_{t+j} matters for *payoffs* — agent is risk averse — but with quadratic utility volatility doesn't matter for optimal policy

Consumption dynamics

- Change in consumption

$$\Delta c_t \equiv c_t - c_{t-1} = c_t - \mathbb{E}_{t-1}\{c_t\} = \frac{r}{1+r}\left(w_t - \mathbb{E}_{t-1}\{w_t\}\right)$$

driven purely by *innovations to permanent income*

- Since $a_t = \mathbb{E}_{t-1}\{a_t\}$, these innovations are given by

$$w_t - \mathbb{E}_{t-1}\{w_t\} = \sum_{j=0}^{\infty} R^{-j}\left(\mathbb{E}_t - \mathbb{E}_{t-1}\right)\{y_{t+j}\}$$

so that

$$\Delta c_t = \frac{r}{1+r}\sum_{j=0}^{\infty} R^{-j}\left(\mathbb{E}_t - \mathbb{E}_{t-1}\right)\{y_{t+j}\}$$

- In short, changes in consumption are proportional to revisions in expected income due to the arrival of new information

Permanent and transitory shocks

- **Example**: suppose income has a *permanent component* z_t and a *transitory component* u_t as in

$$y_t = z_t + u_t$$

$$z_t = z_{t-1} + \varepsilon_t$$

where the shocks u_t and ε_t are IID over time, independent of each other, and have mean zero

- What are the revisions $(\mathbb{E}_t - \mathbb{E}_{t-1})\{y_{t+j}\}$ for this process?

Revisions to expected income

- For $j = 0$ we have

$$(\mathbb{E}_t - \mathbb{E}_{t-1})\{y_t\} = (\mathbb{E}_t - \mathbb{E}_{t-1})(y_{t-1} + u_t - u_{t-1} + \varepsilon_t)$$

$$= u_t + \varepsilon_t$$

- For $j = 1$ we have

$$(\mathbb{E}_t - \mathbb{E}_{t-1})\{y_{t+1}\} = (\mathbb{E}_t - \mathbb{E}_{t-1})(y_t + u_{t+1} - u_t + \varepsilon_{t+1})$$

$$= u_t + \varepsilon_t + (\mathbb{E}_t - \mathbb{E}_{t-1})(u_{t+1} - u_t + \varepsilon_{t+1})$$

$$= \varepsilon_t$$

Revisions to expected income

- Continuing in the same way

$$(\mathbb{E}_t - \mathbb{E}_{t-1})\{y_{t+j}\} = \varepsilon_t \qquad \text{for any } j \geq 1$$

- Hence for this example

$$\Delta c_t = \frac{r}{1+r}\sum_{j=0}^{\infty} R^{-j}(\mathbb{E}_t - \mathbb{E}_{t-1})\{y_{t+j}\}$$

$$= \frac{r}{1+r}\left(u_t + \varepsilon_t + \sum_{j=1}^{\infty} R^{-j}\varepsilon_t\right)$$

$$= \frac{r}{1+r}\left(u_t + \sum_{j=0}^{\infty} R^{-j}\varepsilon_t\right)$$

Response to permanent and transitory shocks

- This simplifies to

$$\Delta c_t = \varepsilon_t + \frac{r}{1+r} u_t$$

- In this example, consumption responds 1-for-1 to permanent shocks ε_t but is much less responsive to transitory shocks u_t

Saving motives

- Three basic motives for saving/dissaving

  (i) *intertemporal substitution* — β vs R, operates even if y_t is deterministic

  (ii) *consumption smoothing* — desire to smooth consumption over different income shocks, operates even if utility is quadratic

  (iii) *precautionary saving* — insurance against future income risk, need to go beyond certainty equivalence

Precautionary saving

- Two period example

- Single agent with risk averse preferences

$$u(c_0) + \beta\mathbb{E}\{\, u(c_1)\,\}$$

- Budget constraints

$$c_0 + a_1 = y_0, \qquad \text{and} \qquad c_1 = Ra_1 + y_1$$

- Stochastic income $y_1 \sim F(y_1)$

- Choose a_1 to maximize

$$u(y_0 - a_1) = \beta \int u(Ra_1 + y_1)\, dF(y_1)$$

Precautionary saving example

- Suppose $\beta R = 1$, no intertemporal substitution motive

- Consumption Euler equation

$$u'(y_0 - a_1) = \int u'(Ra_1 + y_1)\, dF(y_1)$$

- Since $u''(c) < 0$, LHS strictly increasing in a_1 and RHS strictly decreasing in a_1

- Pins down a_1 and hence $c_0 = y_0 - a_1$

Income risk

- How does saving a_1 respond to greater income risk?

- Consider *mean-preserving spread*. Write $y_1 = \bar{y}_1 + \varepsilon$ with mean $\bar{y}_1$ and mean zero risk $\varepsilon \sim G(\varepsilon)$

- Now write consumption Euler equation

$$u'(y_0 - a_1) = \int u'(Ra_1 + \bar{y}_1 + \varepsilon)\, dG(\varepsilon)$$

- If *marginal utility is convex*, i.e., if $u'''(c) > 0$, then by Jensen's inequality we have

$$\int u'(Ra_1 + \bar{y}_1 + \varepsilon)\, dG(\varepsilon) > u'(Ra_1 + \bar{y}_1)$$

- So if marginal utility is convex, income risk leads to more saving

Prudence

- Risk aversion refers to curvature of utility function $u(c)$.
  'Prudence' refers to curvature of marginal utility function $u'(c)$

- CRRA utility function

$$u(c) = \frac{c^{1-\alpha} - 1}{1 - \alpha}, \qquad \alpha > 0$$

  Risk aversion $u''(c) < 0$ and prudence $u'''(c) > 0$

- Quadratic utility function

$$u(c) = c - \frac{b}{2}c^2, \qquad b > 0$$

  Risk aversion $u''(c) < 0$ but no prudence $u'''(c) = 0$

Dynamic version

- Finite periods $t = 0, 1, ..., T$

- Budget constraints

$$c_t + a_{t+1} = Ra_t + y_t$$

- IID income shocks $y_t \sim F(y_t)$

- Bellman equation

$$V_t(a, y) = \max_{a'} \left[u(Ra + y - a') + \beta \int V_{t+1}(a', y') dF(y') \right]$$

- Finite horizon will let us do backwards induction from $t = T$ given that $a_{T+1} = 0$ so that

$$V_T(a, y) = u(Ra + y)$$

Cash-on-hand

- Define '*cash-on-hand*' from RHS of budget constraint

$$x_t \equiv Ra_t + y_t$$

- Evolves according to

$$x_{t+1} = Ra_{t+1} + y_{t+1} = R(x_t - c_t) + y_{t+1}$$

- Terminal condition can be written

$$V_T(x) = u(x)$$

- So $V_T(x)$ inherits all properties of $u(x)$ and hence exhibits prudence if $u(x)$ does

Backwards induction

- Then for one period earlier

$$V_{T-1}(x) = \max_c \left[u(c) + \beta \int u(R(x-c)+y')dF(y') \right]$$

since $V_T(x') = u(x')$ and $x' = R(x-c)+y'$

- If $u'''(c) > 0$ mean-preserving spread will decrease optimal c, saving increases as insurance against more income risk

- Note $V_{T-1}(x)$ is sum of concave functions hence concave and by envelope theorem

$$V_{T-1}'''(x) = \beta R^3 \int u'''(R(x-c)+y')dF(y') > 0$$

Again, value function inherits key properties of utility function

Backwards induction

- One period even earlier

$$V_{T-2}(x) = \max_c \left[u(c) + \beta \int V_{T-1}(R(x-c) + y')dF(y') \right]$$

- Note $V_{T-2}(x)$ is sum of concave functions hence concave and again

$$V'''_{T-2}(x) = \beta R^3 \int V'''_{T-1}(R(x-c) + y')dF(y') > 0$$

- Iterate all the way back to

$$V_0(x) = \max_c \left[u(c) + \beta \int V_1(R(x-c) + y')dF(y') \right]$$

- At each step of iteration $V_t(x)$ is concave and exhibits prudence